Math 201: Introduction to Probability

Midterm 2

November 19, 2019

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

Instructions:

1. Indicate your instructor with a check in the appropriate box:

Tucker	MW 10:25	
Zhang	MW 14:00	

2. Read the notes below:

   - **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
   - **Notes of any kind are strictly forbidden.**
   - **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
   - **You do not need to simplify complicated numerical expressions such as** $\binom{100}{30}$ **and** $50!$ **to a number.**
   - **You are responsible for checking that this exam has all 13 pages.**

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:______________________________

QUESTION	VALUE	SCORE
1	20	
2	15	
3	10	
4	10	
5	6	
6	12	
7	7	
8	15	
9	5	
TOTAL	100	

1. (20 points) Suppose random variable X has a probability density function

$$f(x) = \begin{cases} \frac{3}{(1+x)^4}, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

(a) Compute the expectation $E(X)$.

Solution

$$\begin{aligned} E(X) &= \int_0^\infty \frac{3x}{(1+x)^4} dx = \int_1^\infty \frac{3(u-1)}{u^4} du \\ &= \int_1^\infty \frac{3}{u^3} - \frac{1}{u^4} du \\ &= \frac{3}{2} - 1 = \frac{1}{2}. \end{aligned}$$

(b) Compute the variance Var(X).

Solution

$$\begin{aligned} E(X^2) &= \int_0^\infty \frac{3x^2}{(1+x)^4} dx = \int_1^\infty \frac{3(u-1)^2}{u^4} du \\ &= \int_1^\infty \frac{3}{u^2} - \frac{6}{u^3} + \frac{3}{u^4} du \\ &= 3 - 3 + 1 = 1 \end{aligned}$$

$$Var(X) = E(X^2) - (EX)^2 = 1 - \frac{1}{4} = \frac{3}{4}.$$

(c) Compute the probability $P(X > 1)$. **Solution**

$$P(X > 1) = \int_1^\infty \frac{3}{(1+x)^4} dx = \int_2^\infty \frac{3}{u^4} du = \frac{1}{8}.$$

(d) Compute the conditional probability $P(X > 1 | X \leq 3)$.

Solution

$$P(X \leq 3) = \int_0^3 \frac{3}{(1+x)^4} dx = \int_1^4 \frac{3}{u^4} du = \frac{63}{64}$$

$$P(1 < X \leq 3) = \int_1^3 \frac{3}{(1+x)^4} dx = \int_2^4 \frac{3}{u^4} du = \frac{7}{64}$$

$$P(X > 1 | X \leq 3) = \frac{P(1 < X \leq 3)}{P(X \leq 3)} = \frac{7}{63} = \frac{1}{9}.$$

2. (15 points) Suppose that the distribution of the lifetime of a car battery, produced by a certain car company, is well approximated by a normal distribution with a mean of 1200 hours and variance 10^4.

(a) What is the approximate probability that a car battery has lifetimes less than 1100 hours? **Solution**

$X \sim N(1200, 10000)$, $\frac{X-1200}{100} \sim N(0,1)$

$$P(X < 1100) = P(\frac{X-1200}{100} < -1) = \Phi(-1) = 1 - \Phi(1) = 1 - 0.8413 = 0.1587$$

(b) Suppose that the probability that a car battery has lifetimes less than t_0 hours is equal to 0.9. What is the approximate probability that a batch of 100 car batteries will contain at least 92 whose lifetimes are less than t_0 hours? **Solution**

S=the number of cars whose lifetimes are less than t_0 hours. Then $S \sim Bin(100, 0.9)$.

Approximation: $\frac{S-90}{\sqrt{100\times 0.9\times 0.1}} = \frac{S-90}{3} \sim N(0,1)$

$P(S \geq 92) = P(\frac{S-90}{3} \geq \frac{2}{3}) = 1 - \Phi(\frac{2}{3}) = 1 - 0.7486 = 0.2514$

3. (10 points) The lifetime of a lightbulb can be modeled with an exponential random variable with an expected lifetime of 1000 days.

(a) Find the probability that the lightbulb will function for more than 1500 days.

Solution

$E(X) = \frac{1}{\lambda} = 1000, \ \lambda = \frac{1}{1000}, \ X \sim Exp(\frac{1}{1000})$.

$P(X > 1500) = e^{-1500\lambda} = e^{-1.5}$.

(b) Find the probability that the lightbulb will function for more than 1500 days, given that it is still functional after 500 days.

Memoryless property: $P(X > 1500|X > 500) = P(X > 1000) = e^{-1000\lambda} = e^{-1}$.

4. (10 points) Let X be a continuous random variable with probability density function $f(x) = \frac{1}{2}e^{-|x|}$. Find the moment generating function $M_X(t)$ of X. (Be careful about the possibility that $M_X(t)$ might be infinite for some values of t).

Solution

We have $E[e^{tX}] = \frac{1}{2}\int_{-\infty}^{\infty} e^{tx+|x|}\,dx$. The integral breaks into two pieces, $\int_0^\infty e^{(t-1)x}\,dx$ and $\int_{-\infty}^0 e^{(t+1)x}\,dx$. Clearly the first integral does not converge for $t \geq 1$ and the second does not converge for $t \leq -1$. For $t \in (-1, 1)$, we have

$$\int_0^\infty e^{(t-1)x}\,dx = \frac{1}{t-1}(0-1) = -\frac{1}{t-1}$$

and

$$\int_{-\infty}^0 e^{(t+1)x}\,dx = \frac{1}{t+1}(1-0) = \frac{1}{t+1}.$$

So we get

$$\frac{1}{2}\left(\frac{1}{t+1} - \frac{1}{t-1}\right) = \frac{1}{1-t^2}$$

5. (6 points) Suppose that Y is a random variable with moment generating function $M_Y(t) = \frac{e^t}{2} + \frac{1}{2}$. Find $E[Y^2]$.

Solution.

We have $E[Y^2] = M_Y''(0) = \frac{1}{2}$.

6. (12 points)

There are four jars of jelly beans. The first one contains one jelly bean, the second contains two jelly beans, the third contains three jelly beans, and the fourth contains four jelly beans (for a total of ten jelly beans).

(a) One of the jars is selected at random. Let X denote the number of jelly beans in the jar that was selected. Find $E[X]$.

Solution.

Each jar has a 1/4 change of being selected so we get

$$E[X] = \frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} = \frac{10}{4} = \frac{5}{2}$$

(b) A jelly bean is selected at random from the 10 jelly beans. Let Y denote the number of jelly beans in the jar containing that jelly bean (including that jelly bean itself). Find $E[Y]$.

Solution

There are ten beans. There is a 1/10 chance the bean is selected from the jar with one bean, a 2/10 chance the bean is selected from the jar with two beans, and so on. We get that $E[Y]$ thus equals

$$1 \cdot \frac{1}{10} + 2 \cdot \frac{2}{10} + 3 \cdot \frac{3}{10} + 4 \cdot \frac{4}{10} = \frac{30}{10} = 3.$$

7. (7 points)

Suppose that the number of accidents at a factory in a month follows a Poisson distribution and that the proportion of months in which there are no accidents is $1/e^2$. Find the chance that there is exactly one accident in a given month at the factory. (You may leave e in your answer.)

Solution

We have $P(X = 0) = e^{-\lambda} = 1/e^2 = e^{-2}$ so the parameter λ for our distribution is 2, where X is the random variable equalling the number of accidents in a month at the factory. Then $P(X = 1) = e^{-2} \cdot \frac{2}{1!} = \frac{2}{e^2}$.

8. (15 points)

Suppose we have random variables X and Y with joint probability mass function summarized below.

		Y			
		0	1	2	3
X	0	$\frac{1}{10}$	$\frac{2}{10}$	0	$\frac{2}{10}$
	1	$\frac{4}{10}$	0	$\frac{1}{10}$	0

This means that $P(X = 0, Y = 0) = \frac{1}{10}$, $P(X = 0, Y = 2) = 0$, $PX(X = 1, Y = 2) = \frac{1}{10}$, and so on.

(a) Find $P(Y \leq 1)$

Solution

We add up $P(X = 0, Y = 0) + P(X = 0, Y = 1) + P(X = 1, Y = 0) + P(X = 1, Y = 1)$ and get

$$\frac{1}{10} + \frac{2}{10} + \frac{4}{10} = \frac{7}{10}.$$

(b) Find $E[X]$.

Solution.

Let's ignore the cells where $X = 0$ since they contribute nothing. So we simply get $1 \cdot \frac{4}{10} + 1 \cdot \frac{1}{10} = \frac{1}{2}$.

As on previous page, we have X and Y with joint probability mass function

		Y			
		0	1	2	3
X	0	$\frac{1}{10}$	$\frac{2}{10}$	0	$\frac{2}{10}$
	1	$\frac{4}{10}$	0	$\frac{1}{10}$	0

(c) Find the conditional probability $P(X = 1 \,|\, Y = 0)$.

Solution

We have $P(X = 1, Y = 0) = \frac{4}{10}$ and $P(Y = 0) = \frac{1}{2}$, so

$$P(X = 1 \,|\, Y = 0) = \frac{P(X = 1, Y = 0)}{P(Y = 0)} = 4/5$$

9. (5 points) Suppose that we have a random variable with the property that $E[(X-1)] = 1$ and $E[X^2 + 2] = 11$. Find $\mathrm{Var}(X)$.

Solution

We have $E[X] = 1 + 1 = 2$ and $E[X^2] = 11 - 2 - 9$. So $\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 5$.