Math 201: Introduction to Probability

Final Exam

October 7, 2021

NAME (please print legibly): ____

Your University ID Number: ____

Instructions:

1. Indicate your instructor with a check in the appropriate box:

Krishnan	MW 10:25	
Chio	MW 14:00	

2. Read the notes below:

   - **The presence of any electronic or calculating device at this exam is strictly forbidden, including (but not limited to) calculators, cell phones, and iPods.**
   - **Notes of any kind are strictly forbidden.**
   - **Show work and justify all answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.**
   - **You do not need to simplify complicated numerical expressions such as** $\binom{100}{30}$ **and** $50!$ **to a number.**
   - **You are responsible for checking that this exam has all 22 pages.**

3. Read the following Academic Honesty Statement and sign:

   I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

   Signature:____

Part A		
QUESTION	VALUE	SCORE
1	10	
2	10	
3	15	
4	10	
5	5	
6	10	
TOTAL	60	

Part B		
QUESTION	VALUE	SCORE
1	10	
2	15	
3	15	
4	30	
5	10	
TOTAL	80	

Part A

1. (10 points) Recall that a standard deck of cards has 52 cards, divided into 4 suits (clubs, diamonds, hearts, spades), each suit having 13 values (A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K). You play a poker game at the casino. From a deck of cards the dealer gives you 5 cards. These 5 cards constitute your hand.

(a) The deck of cards is defective. It does not have ace of spades, and instead has two aces of diamonds. How many different hands can you get?

Solution. case(1): have both ace of diamonds. The other 3 cards can be anything: $\binom{50}{3}$. case(2): exactly one ace of diamonds. The other 4 cards cannot be the other ace of diamonds: $\binom{50}{4}$. case(3): no ace of diamonds. $\binom{50}{5}$. The three cases are mutually exclusive and cover all scenarios, so the total number is

$$\binom{50}{3} + \binom{50}{4} + \binom{50}{5}.$$

(b) This time, the dealer deals out a hand of 5 cards from a normal deck. What is the probability that you have a hand two pairs (*Example:* (Ace hearts, Ace spade, 3 diamonds, 3 hearts, X) where X is any card that is not an ace or 3.)?

Solution.

First we choose the two values for the pairs: $\binom{13}{2}$. Each value has 4 suits, so there are $\binom{4}{2}$ choices. There are now $52 - 4 = 48$ choices for the 5-th card, but we cannot use the 4 cards which have the same value as the pairs, there are $48 - 4 = 44$ choices left, so the total number is

$$44\binom{4}{2}\binom{4}{2}\binom{13}{2}$$

The probability of getting a hand with two pairs is

$$\frac{44\binom{4}{2}\binom{4}{2}\binom{13}{2}}{\binom{52}{5}}$$

2. (10 points) Let $X \sim N(\mu, \sigma^2)$, the normal distribution with mean μ and variance σ^2.

(a) What is the probability that X is within 1 standard deviation from the mean? *Hint: your answer must be a number.*

Solution. By writing $X = \sigma Z + \mu$,

$$\begin{aligned} \mathbb{P}(\mu - \sigma < X < \mu + \sigma) &= \mathbb{P}(\mu - \sigma < \sigma Z + \mu < \mu + \sigma) \\ &= \mathbb{P}(-1 < Z < 1) \\ &= \Phi(1) - \Phi(-1) \\ &= \Phi(1) - (1 - \Phi(1)) \\ &= 2(0.8413) - 1 = 0.6826 \end{aligned}$$

(b) Let $Y \sim N(0, 1)$ be a random variable independent of X. Find $\mathbb{E}[7X(5Y + 1)]$. *Hint: Your final answer must be in terms of μ*

Solution.

$$\begin{aligned} \mathbb{E}[7X(5Y + 1)] &= \mathbb{E}[35XY + 7X] && (1) \\ &= 35\mathbb{E}[XY] + 7\mathbb{E}[X] && (2) \\ &= 35\mathbb{E}[X]\mathbb{E}[Y] + 7\mathbb{E}[X] && (3) \\ &= 35 \cdot \mu \cdot 0 + 7\mu = 7\mu. && (4) \end{aligned}$$

3. (15 points) You want to find out how popular pineapple is on pizzas. You randomly called $90,000$ people around the US and among them $42,000$ said pineapple on a pizza is unacceptable.

(a) Give a 95% confidence interval for the true proportion who find pineapple on a pizza unacceptable.

Solution. By the formula

$$\begin{aligned} 2\Phi(2\epsilon\sqrt{n}) - 1 &= 0.95 \\ \Phi(2\epsilon\sqrt{n}) &= 0.975 \end{aligned}$$

Using the table we find $2\epsilon\sqrt{n} = 1.96$. Plug in $n = 90000$ we solve $\epsilon = 1.96/600 = 0.0033$. The observed proportion is $\hat{p} = \frac{42000}{90000} = \frac{7}{15}$. or 0.4667, so the 95% confidence interval is

$$\begin{aligned} (\hat{p} - \epsilon, \hat{p} + \epsilon) &= (0.4667 - 0.0033, 0.4667 + 0.0033) \\ &= (0.4634, 0.47) \end{aligned}$$

(b) A national vote was held about pineapple on a pizza. The result says 40% of the people find pineapple on a pizza unacceptable. Alfredo's Pizza Cafe sells two kinds of pizza with the following prices

(i)A pizza with pineapple, \$9 ;

(ii)A pizza without pineapple, \$10.

Let X_n be the money Alfredo's Pizza Cafe has from selling n pizzas. Express X_n in terms of a Binomial distribution.

Solution. Let S_n be the number of pizzas sold without pineapple, then $S_n \sim \text{Binom}(n, 0.4)$. Then

$$\begin{aligned} X_n &= 10S_n + 9(n - S_n) \\ &= S_n + 9n. \end{aligned}$$

(c) Find

$$\lim_{n\to\infty} \mathbb{P}(X_n > 9.5n).$$

Solution.

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(X_n > 9.5n) &= \lim_{n\to\infty} \mathbb{P}(S_n + 9n > 9.5n) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{n} > 0.5\right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\frac{S_n}{n} - 0.4 > 0.1\right) \\
&\le \lim_{n\to\infty} \mathbb{P}\left(\left|\frac{S_n}{n} - 0.4\right| > 0.1\right) \\
&= 0
\end{aligned}$$

by Law of Large Numbers.

4. (10 points)

Let $X \sim \text{Geom}(1/5)$, that is, for any positive intger k,

$$\mathbb{P}(X = k) = \left(\frac{4}{5}\right)^{k-1} \cdot \left(\frac{1}{5}\right),$$

and $\mathbb{P}(X = k) = 0$ for other values of k.

To receive full credit on the questions below, you must evaluate any infinite sums.

(a) Find $\mathbb{P}(X > 2)$.

Answer:

(b) Find $\mathbb{P}(X \text{ is even})$.

Answer:

5. (5 points) The stock price of company A can either go up, go down, or run flat tomorrow. Suppose the probability that the price will go up tomorrow is 0.6. If it does not go up, then with probability 0.9 it will go down.

(a) Let A_1 be the event that the stock price will go up, and A_2 be the event it will go down. Fill in the blanks.

Answer:

$$\mathbb{P}(A_1) = \underline{0.6}, \quad \mathbb{P}(A_2|A_1^c) = \underline{0.9}.$$

(b) Fill in the blanks.

Answer:

Re-arranging the law of total probability,

$$\mathbb{P}(A_2 \cap A_1^c) = \mathbb{P}(A_2) - \underline{P(A_2 \cap A_1)}.$$

Combine this with the inclusion-exclusion formula,

$$\begin{aligned} \mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \underline{P(A_1 \cap A_2)} \\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2 \cap A_1^c). \\ &= \mathbb{P}(A_1) + \underline{P(A_2|A_1^c)} \cdot \mathbb{P}(A_1^c) \end{aligned}$$

(c) Use (b) to find the probability that the stock price will run flat tomorrow.

Answer:

The probability is: ______________

Solution. Let A_1 be the event that the stock price will go up, and A_2 be the event it will go down. The information provided are

$$\mathbb{P}(A_1) = 0.6, \quad \mathbb{P}(A_2 | A_1^c) = 0.9.$$

First we find $\mathbb{P}(A_1 \cup A_2)$. By law of total probability,

$$\mathbb{P}(A_2) = \mathbb{P}(A_2 \cap A_1) + \mathbb{P}(A_2 \cap A_1^c).$$

Using this, we find

$$\begin{aligned}
\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2 \cap A_1^c) \\
&= \mathbb{P}(A_1) + \mathbb{P}(A_2 | A_1^c)\mathbb{P}(A_1^c) \\
&= 0.6 + 0.9(0.4) = 0.96.
\end{aligned}$$

The probability that the stock price will run flat tomorrow is

$$\mathbb{P}((A_1 \cup A_2)^c) = 1 - \mathbb{P}(A_1 \cup A_2) = 1 - 0.96 = 0.04.$$

6. (10 points) Let X be a random variable with probability density function

$$f_X(x) = \begin{cases} e^{-x} + 2cx & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant.

(a) Find c.

Solution. Since f_X is a pdf, its integral over the real line is equal to 1.

$$\begin{aligned} \int_0^1 e^{-x} + 2cx\, dx &= 1 \\ [-e^{-x} + cx^2]_0^1 &= 1 \\ (-e^{-1} + c) - (-1 + 0) &= 1 \\ c &= e^{-1}. \end{aligned}$$

(b) Find $\mathbb{E}[X]$.

Solution.

$$\begin{aligned} \mathbb{E}[X] &= \int_0^1 x(e^{-x} + 2e^{-1}x)\, dx \\ &= \int_0^1 xe^{-x} + 2e^{-1}x^2\, dx \\ &= \left[-xe^{-x} - e^{-x} + \frac{2e^{-1}}{3}x^3 \right]_0^1 \\ &= \left(-e^{-1} - e^{-1} + \frac{2e^{-1}}{3} \right) - (0 - 1 + 0) \\ &= 1 - \frac{4}{3e}. \end{aligned}$$

Part B

1. (10 points) In each of the following questions identify the described random variable as either Bernoulli, Binomial, Poisson, Geometric, or Exponential and describe a reasonable numerical value for all its parameters. Please use the conventional notation for the parameters i.e., p, λ, n, etc. No explanation is necessary for this problem. Only give your answers.

1. Two 6-sided dice are simultaneously rolled over and over again. Each time the product of the dice is 6 or 12, someone gives you 1 dollar. Let X_1 be the amount of money you have received after 50 rolls. What type of random variable is X_1 and what are its parameters?

   Answer: Binomial$(50, 2/9)$

   The p parameter for the Binomial comes from the calculation $P(X_1 \cdot X_2 \in \{6, 12\}) = |\{(2,3),(3,2),(6,1),(1,6),(6,2),(2,6),(4,3),(3,4)\}|/36 = 2/9$.

2. Han loves to play basketball in his driveway. On average, he scores a basket about 3 out of 5 attempts. His mother calls him in for dinner, but Han is determined to score one more basket before going inside. Let X_2 be the number of attempts before he comes inside. What type of random variable is X_2 and what are its parameters?

   Answer: $X_2 \sim$ Geometric$(3/5)$

   Assuming his attempts are independent, the probability he makes a basket is $3/5$.

3. An urn contains 1 red, 2 black, and 3 white balls. You choose 2 balls uniformly, without replacement. If you chose the red ball, you win 1 dollar, otherwise you win nothing. Let X_3 be the amount of money you win. What type of random variable is X_3 and what are its parameters?

   Answer: $X_3 \sim$ Bernoulli(p)

   The probability of choosing the red ball is $p = 1 \cdot \binom{5}{1}/\binom{6}{2} = 1/3$

4. Suppose that X_4 is a continous random variable with mean equal to 2 with the property that for any real numbers s and t we have $P(X_4 > s) = P(X_4 > t + s \mid X_4 > t)$ (this is sometimes called the "memoryless" poperty). What type of random variable is X_4 and what are its parameters?

   Answer: $X_4 \sim \text{Exp}(1/2)$

   We have used the fact that $\text{Exp}(\lambda)$ has mean $1/\lambda$.

5. Suppose X_5 represents the number of people coming into a store on a certain day. On average the owner sees 5 customers per day. What type of random variable would you use to model X_5 and what are its parameters?

   Answer: $X_5 \sim \text{Poisson}(5)$

2. (15 points) Suppose (X, Y) are uniformly distributed on the triangular region

$$D = \{(x, y) \colon x + y \leq 1, x \geq 0, y \geq 0\}$$

.

1. Find the marginal density function $f_X(t)$ of X:

$$f_X(s) = \int_{-infty}^{\infty} f_{XY}(s, t)dt = \int_0^{1-s} 2dt = 2(1 - s)$$

Answer:

2. Find $M_X(t)$.

$$\begin{aligned} M_X(t) &= \int_0^1 2(1 - s)e^{st} ds \\ &= 2\left[\left[\frac{e^{st}}{t}\right]_0^1 - se^{st}t\Big|_0^1 + \int_0^1 \frac{e^{st}}{t} ds\right] \\ &= 2\left[\frac{(e^t - 1)}{t} - \frac{e^t}{t} + \frac{e^t - 1}{t^2}\right] \end{aligned}$$

Answer:

3. Find the correlation coefficient $\mathrm{Corr}(X, Y)$

$$\begin{aligned}\mathbb{E}[XY] &= \frac{1}{12} \\ \mathbb{E}[X] = \mathbb{E}[Y] &= \frac{1}{3} \\ \mathbb{E}[X^2] = \mathbb{E}[Y^2] &= \frac{1}{6} \\ \mathrm{Corr}(X, Y) = -\frac{1}{2}\end{aligned}$$

Answer:

4. Determine whether X and Y are independent or not. Mathematically justify your answer.

Solution: If X and Y were independent, then $f_X(s)f_Y(y) = f_{XY}(s,t)$ for all s, t. By reflection symmetry of the triangle across the line $y = x$, we see that $f_Y(s) = f_X(s)$.

Clearly $f_X(3/4) = 2(1 - 3/4) = 1/2$. However, $f_{XY}(3/4, 3/4) = 0$ since $(3/4, 3/4)$ lies outside of the triangle. Therefore

$$f_X(3/4)f_Y(3/4) \neq f_{XY}(3/4, 3/4),$$

and shows that they are not independent.

3. (15 points) Suppose X and Y are independent random variables.

1. Show that

$$\mathbb{E}[e^{sX+tY}] = \mathbb{E}[e^{sX}]\mathbb{E}[e^{tY}]$$

Solution: Using independence of X and Y we know that $\mathbb{E}[f(X)g(Y)] = \mathbb{E}[f(X)]\mathbb{E}[g(Y)]$. Thus

$$\begin{aligned}\mathbb{E}[e^{sX+tY}] &= \mathbb{E}[e^{sX}e^{tY}] \\ &= \mathbb{E}[e^{sX}]\mathbb{E}[e^{tY}]\end{aligned}$$

Answer:

2. Suppose X and Y are independent, and

$$\mathbb{E}[e^{sX+tY}] = \frac{1}{3}e^{-s+2t} + \frac{1}{6}e^{-s+3t} + \frac{1}{6}e^{2s+2t} + \frac{1}{3}e^{2s+3t}$$

find the joint pmf of X and Y and enter it in the table below.

There was a typo in the above mgf, it should have been

$$\frac{1}{3}e^{-s+2t} + \frac{1}{6}e^{-s+3t} + \frac{1}{6}e^{2s+3t} + \frac{1}{3}e^{2s+2t}$$

So, whichever covariance you calculated (one that is consistent with the mgf given), or one that is consistent with the fact that X and Y are independent will get you points.

X \ Y	2	3
-1	$\frac{1}{3}$	$\frac{1}{6}$
2	$\frac{1}{6}$	$\frac{1}{3}$

3. Compute $\text{Cov}(X, Y)$.

$$\begin{aligned} \mathbb{E}[X] &= \frac{1}{2} \\ \mathbb{E}[Y] &= \frac{5}{2} \\ \mathbb{E}[XY] &= \frac{3}{2} \\ \text{Cov}(X, Y) &= \frac{1}{4} \end{aligned}$$

4. (30 points)

A computer store manager has employed us as her analyst. She has a fixed stock of 1900 items in her store for the year, coming from the manufacturer. She has tasked us with estimating the probability that she will run out of stock this year, so that she can decide whether or not to order extra stock. On average, they see around 1825 customers a year.

Number the days of the year $\{1, \dots, 365\}$, and let X_i be the number of customers buying a computer on day i for $i = 1, \dots, 365$. Assume that $X_i \sim \text{Poisson}(\lambda)$, where λ is an unknown parameter that is to be determined. It is also reasonable to assume that $\{X_1, \dots, X_{365}\}$ are independent, since if a customer arrives on day 5, it's unlikely that they will come again on day 20. Let

$$S = X_1 + X_2 + \cdots + X_{365}$$

represent the total number of customers arriving in the year.

1. What is λ, the Poisson parameter of X_i? *Hint: find the mean of S.*
   Solution:

$$\begin{aligned}\mathbb{E}[S] &= 365\mathbb{E}[X_1] = 365\lambda = 1825 \\ \lambda &= 5\end{aligned}$$

Answer:

2. What is the moment generating function of S?

$$\begin{aligned}\mathbb{E}[e^{tS}] &= \mathbb{E}[e^{tX_1}]^{365} \\ &= e^{1825(e^t-1)}\end{aligned}$$

Answer:

3. What is the distribution of S? You can either give its pdf, or if it is a standard distribution, you can name it with the correct parameter.
Solution:

$S \sim$ Poisson(1825)

$$P(S = k) = \frac{(1825)^k e^{-1825}}{k!}$$

Answer:

4. What is the *exact* probability that (strictly) more than 1900 people come to the store and buy a computer that year? Your answer can be an infinite sum.
Solution:

$$P(S > 1900) = \sum_{k=1901}^{\infty} \frac{e^{-1825}(1825)^k}{k!}$$

Answer:

5. What is the variance of S?

Solution:

$$\mathrm{Var}(S) = 1825$$

Answer:

6. Numerically estimate the probability that $P(S > 1900)$. State any theorems that you used for your approximation.

Solution:

$$P\left(\frac{S - 1825}{\sqrt{1825}} > \frac{1900 - 1825}{\sqrt{1825}}\right) \approx 1 - \Phi\left(\frac{75}{\sqrt{1825}}\right)$$

$$1 - \Phi(1.756)$$

Answer:

5. (10 points) Let X_i be the amount of money earned by a restaurant on Park Avenue on day i.

(a) Given that the owner only knows that $E[X_i] = \$4000$, give the best possible upper bound for the probability that the restaurant will earn at least \$5000 tomorrow.

Solution:

$$P(X > 5000) \leq \frac{\mathbb{E}[X_i]}{5000} = \frac{4}{5}$$

Answer:

(b) Answer part (a) again with the extra knowledge that $Var(X_i) = \$3000$.

Solution:

$$P(X_i - 4000 > 1000) \leq P\left(|X_i - 4000| > 1000\right) \leq \frac{\mathrm{Var}(X_i)}{1000^2} = \frac{3000}{1000^2}$$

Answer: