MTH 201

Final Exam – AM version

May 4, 2020

Name: Key

UR ID: ______________________________

Circle your Instructor's Name:

Ian Alevy Mark Herman

Instructions:

• THIS IS NOT TRUE .... The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• For each problem, please put your final answer in the answer box. We will judge your work outside the box as well (unless specified otherwise) so you still need to show work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$ and you can leave your answers in terms of $\binom{n}{k}$ or $k!$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Summations must also be evaluated, in particular, the symbols "$\sum$" or "$\cdots$" should not appear in final answers.

• This exam is out of 50 points. You are responsible for checking that this exam has all 10 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

____

____

____

YOUR SIGNATURE: ______________________________

Part A		
QUESTION	VALUE	SCORE
1	6	
2	8	
3	8	
4	10	
5	10	
6	8	
TOTAL	50	

Part B		
QUESTION	VALUE	SCORE
1	6	
2	8	
3	6	
4	10	
5	10	
6	10	
TOTAL	50	

Part A

1. (6 points) A bin has 100 apples, 2 of which are rotten. Apples are removed one-by-one from the bin and not replaced.

(a) Find the probability that on draw 4 a rotten apple is removed from the bin.

(b) Find the probability that 1 rotten apple is removed on draws $1, 2, \ldots 49$ and that the 2nd rotten apple is removed on draw 50.

a) gggr
ggrr
grgr
rggr

$$\text{Prob} = \frac{98 \cdot 97 \cdot 96 \cdot 2 + 98 \cdot 97 \cdot 2 \cdot 3}{100 \cdot 99 \cdot 98 \cdot 97} = \frac{1}{50} = .02$$

b) Prob =

$$\text{Prob} = \frac{\binom{2}{1}\binom{98}{48}}{\binom{100}{49}} \cdot \frac{1}{51} = .009899$$

2. (8 points) Let $X \sim \text{Bern}(1/3)$ and $G \sim \text{Geom}(1/4)$. Assume that X and G are independent. We are conducting an experiment in which we would like to measure X but due to the presence of noise in the signal can only measure $O = X + G$. Find the conditional probability

$$P(X = 1|O > 5).$$

$$= \frac{P(O>5|X=1)\,P(X=1)}{P(O>5|X=1)\,P(X=1) + P(O>5|X=0)\,P(X=0)}$$

$$= \frac{P(G>4)\,P(X=1)}{P(G>4)\,P(X=1) + P(G>5)\,P(X=0)}$$

$$= \frac{\left(\frac{3}{4}\right)^4\left(\frac{1}{3}\right)}{\left(\frac{3}{4}\right)^4\left(\frac{1}{3}\right) + \left(\frac{3}{4}\right)^5\left(\frac{2}{3}\right)} = \frac{1}{1+\frac{3}{2}} = \boxed{\frac{2}{5}}$$

3. (8 points) In the town of Cloudchester, 30% of cars are red and 20% of cars are sports cars. If 15% of cars are red sports cars what is the probability that a randomly chosen car is neither red nor a sports car?

$A = \{\text{cars red}\}$ $\quad P(A) = .3$ $\quad P(A \cap B) = .15$

$B = \{\text{sports cars}\}$ $\quad P(B) = .2$

$$P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B)$$

$$= 1 - (P(A) + P(B) - P(A \cap B)) = 1 - (.5 - .15) = \boxed{.65}$$

4. (10 points) Each day Dr. Wiley performs the following experiment: he flips a fair coin repeatedly until he gets tails and counts the number of coin flips needed.

(a) Approximate the probability that in a year there are at least 3 days when he needed more than 10 coin flips. Use either the normal or the Poisson approximation, whichever is more appropriate (and justify your choice). Assume a year has 365 days.

X = days with > 10 flips

$p = \left(\frac{1}{2}\right)^{10}$ $\quad X \sim \text{Binom}\left(365, \left(\frac{1}{2}\right)^{10}\right)$

$np(1-p) = .36$

$np^2 = .00035$ $\Rightarrow$ Poisson is appropriate $\lambda = np = 365\left(\frac{1}{2}\right)^{10}$

$P(X \geq 3) = 1 - P(X<3) = 1 - \left(P(X=0) + P(X=1) + P(X=2)\right)$

$= 1 - \left(e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2} e^{-\lambda}\right) = \boxed{.005791}$

with $\lambda = np = 365\left(\frac{1}{2}\right)^{10} = .356$

(b) Approximate the probability that in a year there are more than 50 days when he needed exactly 3 coin flips. Use either the normal or the Poisson approximation, whichever is more appropriate (and justify your choice).

$p = \left(\frac{1}{2}\right)^3$ $\quad X \sim \text{Binom}\left(365, \frac{1}{8}\right)$

$np(1-p) = 39.92$ $\Rightarrow$ normal appropriate

$np^2 = 5.70$

$P(X > 50) \approx P\left(Z > \frac{50 - np}{\sqrt{np(1-p)}}\right)$ where $Z \sim N(0,1)$

$= P\left(Z > \frac{50 - 365\left(\frac{1}{8}\right)}{\sqrt{365\left(\frac{1}{8}\right)\left(\frac{7}{8}\right)}}\right) = P(Z > .6924)$

$= 1 - \Phi(.6924) = \boxed{.2443}$

if the continuity correction is used get

$P\left(Z > \frac{50.5 - np}{\sqrt{np(1-p)}}\right) = P(Z > .7716) = .2202$

5. (10 points) Let X be a random variable with distribution $X \sim Exp(\lambda)$. Suppose the lifetime of a certain device (measured in months), is given by $Y = \sqrt{X}$.

(a) Compute the cumulative distribution function for Y.

Hint: It will help to recall the cumulative distribution function of X.

$$F_Y(s) = P(Y \le s) = P(\sqrt{X} \le s) = P(X \le s^2) = 1 - e^{-\lambda s^2}$$

$$F_Y(s) = \begin{cases} 0 & s < 0 \\ 1 - e^{-\lambda s^2} & s \ge 0. \end{cases}$$

(b) Compute the probability density function of Y.

$$f_Y(s) = F'(s) = 2\lambda s e^{-\lambda s^2}$$

$$f_Y(s) = \begin{cases} 0 & s < 0 \\ 2s\lambda e^{-\lambda s^2} & s \ge 0. \end{cases}$$

(c) Find the probability the device lasts at least 6 months.

$$P(Y \ge 6) = e^{-\lambda 6^2} = e^{-36\lambda}$$

(d) Find the probability that at least 1 out of 20 devices of this type will last at least 6 months. Assume the 20 are chosen independently.

$$P(\text{at least } 1) = 1 - P(\text{none}) = 1 - \left(P(Y<6)\right)^{20} =$$

$$= 1 - \left(1 - e^{-36\lambda}\right)^{20}$$

6. (8 points) In each of the following questions identify the described random variable as either binomial, Poisson, geometric, or exponential and describe a reasonable numerical value for all its parameters. Please use the conventional notation for the parameters i.e., p, λ, n, etc. In some cases more than one random variable type might fit, but make sure that you select a choice where you can describe numerical values for all the parameters. No explanation is necessary for this problem. Only give your answers.

(a) Two 6-sided dice are simultaneously rolled over and over again. Each time the sum of the two dice is 7 or 11, someone gives you 1 dollar. let X_1 be the amount of money you have received after 50 rolls of both dice. What type of random variable is X_1 and what are its parameters?

$7 = 1,6$ $\quad$ $11 = 5,6$
$2,5$
$3,4$

$$X_1 \sim \text{Bin}\left(50, \frac{8}{36}\right)$$

(b) Lola loves to play basketball in her driveway. On average, she scores a basket about 1 out of 5 attempts. Her mother calls her in for dinner, but Lola is determined to score one more basket before going inside. Let X_2 be the number of attempts before she comes inside. What type of random variable is X_2 and what are its parameters?

$$X_2 \sim \text{Geom}\left(\frac{1}{5}\right)$$

(c) The average time between accidents at the corner of Elmwood Avenue and Monroe Avenue is 2 days. An accident just occurred. Let X_3 be the amount of time, measured in days, until the next accident at the corner. Assuming X_3 is a continuous variable, what is the most convenient choice for the type of random variable X_3 and what are its parameters?

$$X_3 \sim \text{Exp}\left(\frac{1}{2}\right)$$

(d) Referring to part (c), let X_4 be the number of accidents at the corner of Elmwood Avenue and Monroe Avenue in a week. What type of random variable is X_4 and what are its parameters?

$$X_4 \sim \text{Poisson}\left(\frac{7}{2}\right)$$

Part B

1. (6 points) Suppose X and Y are independent random variables with moment generating functions

$$M_X(t) = \frac{1}{2}e^{-t} + \frac{1}{3} + \frac{1}{6}e^{2t}$$
$$M_Y(t) = \frac{1}{3}e^{-t} + \frac{2}{3}e^{t}.$$

Let $Z = X + Y$.

(a) Compute the p.m.f. of Z.

(b) Compute $E[Z^3]$.

a) $M_X M_Y = \left(\frac{1}{2}e^{-t} + \frac{1}{3} + \frac{1}{6}e^{2t}\right)\left(\frac{1}{3}e^{-t} + \frac{2}{3}e^{t}\right)$

$= \frac{1}{6}e^{-2t} + \frac{1}{3} + \frac{1}{9}e^{-t} + \frac{2}{9}e^{t} + \frac{1}{18}e^{t} + \frac{1}{9}e^{3t}$

$= \frac{1}{6}e^{-2t} + \frac{1}{9}e^{-t} + \frac{1}{3} + \frac{5}{18}e^{t} + \frac{1}{9}e^{3t} = M_Z(t)$

K	-2	-1	0	1	2	3
$P_Z(K)$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{5}{18}$	0	$1/9$

b) $E[Z^3] = M_Z^{(3)}(0) =$

$\left(\frac{1}{6}(-2)^3 e^{-2t} + \frac{1}{9}(-1)^3 e^{-t} + \frac{5}{18}e^{t} + \frac{1}{9}3^3 e^{3t}\right)\Big|_{t=0}$

$= \frac{-8}{6} - \frac{1}{9} + \frac{5}{18} + \frac{27}{9} = \boxed{\frac{11}{6}}$

2. (8 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed exponential random variables with rate λ. Let

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(a) Compute the moment generating function of Y_n.

(b) Compute $E[Y_n^3]$.

a) $M_{X_1}(t) = \dfrac{\lambda}{\lambda - t}$

$$M_{Y_n} = E\left[e^{\frac{t}{n}\sum_{i=1}^{n} X_i}\right] = E\left[e^{\frac{t}{n}X_1} e^{\frac{t}{n}X_2} \cdots e^{\frac{t}{n}X_n}\right]$$

$$\overset{\text{independence}}{=} E\left[e^{\frac{t}{n}X_1}\right]\cdots E\left[e^{\frac{t}{n}X_n}\right] = \left(E\left[e^{\frac{t}{n}X_1}\right]\right)^n$$

$$= \left(\frac{\lambda}{\lambda - \frac{t}{n}}\right)^n = \boxed{\left(\frac{n\lambda}{n\lambda - t}\right)^n = M_{Y_n}(t)}$$

b) $$E[Y_n^3] = M_{Y_n}^{(3)}(0) = \left(\frac{d}{dt}\right)^3 \left(\frac{n\lambda}{n\lambda - t}\right)^n \Bigg|_{t=0}$$

$$= (n\lambda)^n \left(\frac{d}{dt}\right)^3 (n\lambda - t)^{-n}\Big|_{t=0}$$

$$= (n\lambda)^n (-n)\left(\frac{d}{dt}\right)^2\left((n\lambda - t)^{-n-1}(-1)\right) = (n\lambda)^n (n)(n+1)\frac{d}{dt}\left((n\lambda - t)^{-n-2}\right)\Big|_{t=0}$$

$$= (n\lambda)^n (n)(n+1)(n+2)(n\lambda - t)^{-n-3}\Big|_{t=0}$$

$$= \frac{(n\lambda)^n n(n+1)(n+2)}{(n\lambda)^{n+3}} = \boxed{\frac{(n+1)(n+2)}{n^2\lambda^3}}$$

3. (6 points) Statisticians estimate that X, the number of patients that arrive at a hospital on a given day, is a random variable with mean 300 and variance 25^2. If more than 600 patients arrive at a hospital on a particular day then the hospital capacity is overwhelmed. The statisticians do not know the distribution of X but they suspect that it is **not** normally distributed.

(a) Give the best possible upper bound on the probability that the hospital is overwhelmed on Friday.

(b) If fewer than 50 patients arrive on a particular day then the hospital loses money and must furlough staff. Give the best possible upper bound on the probability that the hospital furloughs staff on Friday.

a) Chebyshev: $P(X \geq c + \mu) \leq \frac{\sigma^2}{c^2}$

$\cancel{P(X \geq 300 + 300) \leq \frac{25^2}{300^2} = \frac{1}{144}}$

$P(X \geq 601) = P(X \geq 301 + \mu) \leq \frac{25^2}{301^2} = \boxed{.006898}$

b) $P(X \leq \mu - c) \leq \frac{\sigma^2}{c^2}$ also by Chebyshev

$P(X \leq 49) = P(X \leq \mu - 251) \leq \frac{25^2}{251^2} = \boxed{.009920}$

4. (10 points)

(a) Show that if $X \sim N(\mu, \sigma^2)$, then $P(X > 0) = \Phi(\frac{\mu}{\sigma})$ where Φ is the c.d.f. of the standard normal distribution.

$X \overset{(d)}{=} \sigma Z + \mu$ where $Z \sim N(0,1)$

$$P(X>0) = P(\sigma Z + \mu > 0) = P\left(Z > \frac{-\mu}{\sigma}\right) = 1 - P\left(Z \le \frac{-\mu}{\sigma}\right)$$

$$= 1 - \Phi\left(\frac{-\mu}{\sigma}\right) = 1 - \left(1 - \Phi\left(\frac{\mu}{\sigma}\right)\right) = \Phi\left(\frac{\mu}{\sigma}\right)$$

(b) Civil engineers believe that W, the amount of weight (in the unit of 1000) pounds that a certain span of a bridge can withstand without sustaining structural damage is normally distributed with mean 100 and variance (10^2). Suppose the weight of a car is a random variable with mean 3 and variance $(0.3)^2$. Approximately how many cars would have to be on the bridge span for the probability of structural damage to exceed 0.1?

Hint: Use the central limit theorem. Let $X_1, \ldots, X_n$ be independent and identically distributed with mean 3 and variance $(0.3)^2$. Structural damage corresponds to the event $\{S_n - W > 0\}$ where $S_n = X_1 + \cdots + X_n$.

$S_n - W$ has mean $3n - 100$ and variance $.09n + 100$

by CLT $S_n - W \approx N(3n - 100, .09n + 100)$

$$P(S_n - W > 0) =$$

$$= \Phi\left(\frac{3n - 100}{\sqrt{.09n + 100}}\right) > .1$$

$$\boxed{29 \text{ cars}}$$

$$\frac{3n - 100}{\sqrt{.09n+100}} = -1.29 \Rightarrow \begin{array}{l} 3n - 100 = -1.29\sqrt{.09n+100} \\ (3n-100)^2 = 1.29^2\,(.09n+100) \\ n = 28.98 \text{ or } 37.71 \end{array}$$

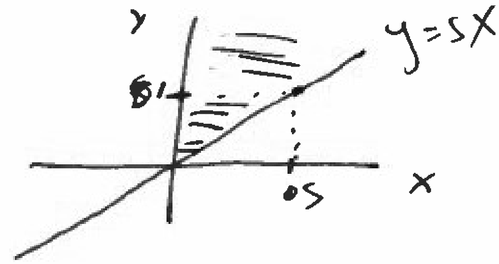
take smaller solution

5. (10 points) Let X and Y be jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} e^{-x-y}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

and let $Z = X/Y$.

(a) Find the cumulative distribution function of Z.



$$P(Z \le s) = P\left(\frac{X}{Y} \le s\right) = P(X \le sY)$$

$$= \int_0^\infty \int_0^{sy} e^{-x-y}\,dx\,dy = \int_0^\infty e^{-y}\left(-e^{-x}\right)\Big|_{x=0}^{sy}\,dy$$

$$= \int_0^\infty e^{-y}\left(1 - e^{-sy}\right)dy = \int_0^\infty e^{-y} - e^{-y(1+s)}\,dy)$$

$$= -e^{-y}\Big|_0^\infty + \frac{e^{-y(1+s)}}{1+s}\Big|_{y=0}^{\infty} = 1 + 0 - \frac{1}{1+s}$$

$$= \frac{1+s-1}{1+s} = \boxed{\frac{s}{s+1}}$$

$$\boxed{F_Z(s) = \begin{cases} 0 & s < 0 \\ \frac{s}{s+1} & s \ge 0. \end{cases}}$$

(b) Find the density function of Z.

$$f_Z(s) = \frac{d}{ds} P(Z \le s) = \frac{d}{ds}\left(\frac{s}{s+1}\right) = \frac{s+1-s}{(s+1)^2} = \frac{1}{(s+1)^2}$$

$$f_Z(s) = \begin{cases} 0 & s < 0 \\ \frac{1}{(1+s)^2} & s \ge 0. \end{cases}$$

6. (10 points) Suppose we roll a fair, 6-sided die repeatedly. Let A_k be the indicator random variable for the event that the kth roll is a 1. Similarly, let B_k be the indicator random variable for the event that the kth roll is a 2.

(a) Find $E[A_i B_j]$ (consider both cases $i = j$ and $i \neq j$).

$i=j$: $E[A_i B_i] = 0$ (can't ~~both~~ be 1 and 2)

$i \neq j$ $E[A_i B_j] = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$

$= P(i\text{th roll } 1,\ j\text{th roll } 2)$

(b) Let X be the number of 1's and Y the number of 2's that occur in n rolls of the die. Find $E[XY]$. Hint: part (a) may help.

$\left(\begin{matrix} X_i = A_i \\ Y_i = B_i \end{matrix} \right)$

$X = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} A_i \quad Y = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} B_i$

$E[XY] = E\left[\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)\right] = \sum_{i=1}^{n} \sum_{j=1}^{n} E[X_i Y_j]$

$= (n^2 - n)\frac{1}{36} = \frac{n(n-1)}{36}$

(c) Find $cov(X, Y)$.

$Cov(X,Y) = E[XY] - E[X]\,E[Y]$

$E[X] = \frac{n}{6} \qquad E[Y] = \frac{n}{6}$

$Cov(X,Y) = \frac{n(n-1)}{36} - \frac{n^2}{36} = \frac{-n}{36}$