MTH 201

Final Exam – PM version

May 4, 2020

Name: Key

UR ID: ______________________

Circle your Instructor's Name:

Ian Alevy   Mark Herman

Instructions:

• THIS IS NOT TRUE .... The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• For each problem, please put your final answer in the answer box. We will judge your work outside the box as well (unless specified otherwise) so you still need to show work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$ and you can leave your answers in terms of $\binom{n}{k}$ or $k!$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Summations must also be evaluated, in particular, the symbols "$\sum$" or "$\cdots$" should not appear in final answers.

• This exam is out of 50 points. You are responsible for checking that this exam has all 10 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE: ______________________

Part A		
QUESTION	VALUE	SCORE
1	6	
2	8	
3	8	
4	10	
5	10	
6	8	
TOTAL	50	

Part B		
QUESTION	VALUE	SCORE
1	6	
2	8	
3	6	
4	10	
5	10	
6	10	
TOTAL	50	

Part A

1. (6 points) A bin has 150 apples, 3 of which are rotten. Apples are removed one-by-one from the bin and not replaced.

(a) Find the probability that on draw 4 an apple which is not rotten is removed from the bin.

(b) Find the probability that 2 rotten apples are removed on draws $1, 2, \ldots 99$ and that the 3rd rotten apple is removed on draw 100.

a) gggg

rggg $\Rightarrow$ 3 ways

rrgg $\Rightarrow$ 3 ways

rrr g

$$\text{Prob} = \cancel{\frac{147 \cdot 146 \cdot 145 \cdot 144 + 3 \cdot 3 \cdot 1}{150 \cdot 149 \cdot 148 \cdot 147}}$$

$$\text{Prob} = \frac{147 \cdot 146 \cdot 145 \cdot 144 + 3 \cdot 3 \cdot 147 \cdot 146 \cdot 145 + 3 \cdot 2 \cdot 147 \cdot 146 \cdot 3 + 3 \cdot 2 \cdot 1 \cdot 147}{150 \cdot 149 \cdot 148 \cdot 147}$$

$$= .98$$

b) $$\frac{\binom{3}{2}\binom{147}{97}}{\binom{150}{99}} \times \frac{1}{51} = .008799$$

2. (8 points) Let $X \sim \text{Bern}(3/4)$ and $G \sim \text{Exp}(1/3)$. Assume that X and G are independent. We are conducting an experiment in which we would like to measure X but due to the presence of noise in the signal can only measure $O = X + G$. Find the conditional probability

$$P(X = 0 | O > 4).$$

$$= \frac{P(O>4|X=0)\,P(X=0)}{P(O>4|X=0)\,P(X=0) + P(O>4|X=1)\,P(X=1)}$$

$$= \frac{P(G>4)\,P(X=0)}{P(G>4)\,P(X=0) + P(G>3)\,P(X=0)}$$

$$= \frac{e^{-4/3}\left(\frac{1}{4}\right)}{e^{-4/3}\left(\frac{1}{4}\right) + e^{-1}\left(\frac{3}{4}\right)} = \boxed{\frac{1}{1+3e^{1/3}}}$$

3. (8 points) In the town of Cloudchester, 30% of dogs have brown fur and 50% of dogs have curly fur. If 40% of dogs have fur which is neither curly nor brown, what is the probability that a randomly chosen dog has fur which is both brown and curly?

$A = \{\text{brown}\}$ & $B = \{\text{curly}\}$

$P(A) = .3$ $P(B) = .5$

$.4 = P(A^c \cap B^c) = P((A \cup B)^c)$

$P(A \cup B) = 1 - .4 = .6$

$\cancel{P(A \cap B) =}$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= .8 - .6 = \boxed{.2}$

4. (10 points) Each day Dr. Wiley performs the following experiment: he rolls a fair, 6-sided die repeatedly until he gets either a 1 or a 2 and counts the number of rolls needed.

(a) Approximate the probability that in a year there are at least 7 days when he needed more than 8 rolls. Use either the normal or the Poisson approximation, whichever is more appropriate (and justify your choice). Assume a year has 365 days.

X = days with more than 8 rolls. $X \sim Bin(365, p)$

$$p = \left(\frac{4}{6}\right)^8$$

$$np(1-p) = 13.69 \Rightarrow \text{normal appropriate}$$

$$np^2 = .56$$

$$P(X \geq 7) \approx P\left(Z \geq \frac{7-np}{\sqrt{np(1-p)}}\right) \quad \text{where } Z \sim N(0,1)$$

$$= P(Z \geq -1.9575) = 1 - \Phi(-1.9575) = 1-(1-\Phi(1.9575))$$

$$\boxed{= .9748}$$

if continuity correction used

$$P(X \geq 6.5) \approx P\left(Z \geq \frac{6.5-np}{\sqrt{np(1-p)}}\right) = P(Z \geq -2.0927) = .9818$$

(b) Approximate the probability that in a year there are more than 7 days when he needed exactly 10 rolls. Use either the normal or the Poisson approximation, whichever is more appropriate (and justify your choice).

$$p = \left(\frac{4}{6}\right)^9 \left(\frac{2}{6}\right) = .00867$$

$$np(1-p) = 3.13$$

$$np^2 = .027 \quad \Rightarrow \text{Poisson appropriate}$$

$$P(X > 7) = 1 - P(X \leq 7) \simeq 1 - P(Y \leq 7) \quad \text{where } Y \sim \text{Poisson}(\lambda)$$

and $\lambda = np = 3.1648$

$$= 1 - \left(e^{-\lambda} + e^{-\lambda}\lambda + e^{-\lambda}\frac{\lambda^2}{2} + e^{-\lambda}\frac{\lambda^3}{3!} + e^{-\lambda}\frac{\lambda^4}{4!} + e^{-\lambda}\frac{\lambda^5}{5!} + e^{-\lambda}\frac{\lambda^6}{6!} + e^{-\lambda}\frac{\lambda^7}{7!}\right)$$

$$= \boxed{.01587}$$

5. (10 points) Let X be a random variable with distribution $X \sim Exp(\lambda)$. Suppose the lifetime of a certain device (measured in months), is given by $Y = X^{1/3}$.

(a) Compute the cumulative distribution function for Y.

Hint: It will help to recall the cumulative distribution function of X.

$$F_Y(s) = P(Y \leq s) = P(X^{1/3} \leq s) = P(X \leq s^3) = 1 - e^{-\lambda s^3}$$

$$F_Y(s) = \begin{cases} 0 & s < 0 \\ 1 - e^{-\lambda s^3} & s \geq 0. \end{cases}$$

(b) Compute the probability density function of Y.

$$f_Y(s) = F_Y'(s) = \frac{d}{ds}\left(1 - e^{-\lambda s^3}\right) = 3\lambda s^2 e^{-\lambda s^3}$$

$$f_Y(s) = \begin{cases} 0 & s < 0 \\ 3\lambda s^2 e^{-\lambda s^3} & s \geq 0. \end{cases}$$

(c) Find the probability the device lasts at least 3 months.

$$P(Y > 3) = e^{-\lambda (3)^3} = e^{-27\lambda}$$

(d) Find the probability that at least 1 out of 10 devices of this type will last at least 3 months. Assume the 10 are chosen independently.

$$P(\text{at least } 1) = 1 - P(\text{none}) = 1 - \left(P(Y \leq 3)\right)^{10}$$

$$= 1 - \left(1 - e^{-27\lambda}\right)^{10}$$

6. (8 points) In each of the following questions identify the described random variable as either binomial, Poisson, geometric, or exponential and describe a reasonable numerical value for all its parameters. Please use the conventional notation for the parameters i.e., p, λ, n, etc. In some cases more than one random variable type might fit, but make sure that you select a choice where you can describe numerical values for all the parameters. No explanation is necessary for this problem. Only give your answers.

(a) Bill loves to play basketball in his driveway. On average, he scores a basket about 1 out of 4 attempts. His mother calls him in for dinner, but Bill is determined to score one more basket before going inside. Let X_1 be the number of attempts before he comes inside. What type of random variable is X_1 and what are its parameters?

$X_1 \sim \text{Geom}\left(\frac{1}{4}\right)$

(b) On average, there are 4 accidents per week at the corner of Elmwood Avenue and Monroe Avenue. Let X_2 be the number of accidents at the corner of Elmwood Avenue and Monroe Avenue in a week. What type of random variable is X_2 and what are its parameters?

$X_2 \sim \text{Poisson}(4)$

(c) This question refers to part (b). An accident just occurred at the Elmood/Monroe corner. Let X_3 be the amount of time, measured in days, until the next accident at the corner. Assuming X_3 is a continuous variable, what is the most convenient choice for the type of random variable X_3 and what are its parameters?

$X_3 \sim \text{Exp}\left(\frac{7}{4}\right)$

(d) A fair coin is flipped repeatedly. Each time the coin is heads, someone gives you 1 dollar. Let X_4 be the amount of money you have received after 20 flips of the coin. What type of random variable is X_4 and what are its parameters?

$X_4 \sim \text{Bin}\left(20, \frac{1}{2}\right)$

Part B

1. (6 points) Suppose X and Y are independent random variables with moment generating functions

$$M_X(t) = \frac{2}{3}e^{-t} + \frac{1}{4} + \frac{1}{12}e^{2t}$$
$$M_Y(t) = \frac{3}{4}e^{-t} + \frac{1}{4}e^{t}.$$

Let $Z = X + Y$.

(a) Compute the p.m.f of Z.

(b) Compute $E[Z^3]$.

a) $M_Z(t) = M_X(t)M_Y(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{4} + \frac{1}{12}e^{2t}\right)\left(\frac{3}{4}e^{-t} + \frac{1}{4}e^{t}\right)$

$= \frac{1}{2}e^{-2t} + \frac{1}{6} + \frac{3}{16}e^{-t} + \frac{1}{16}e^{t} + \frac{1}{16}e^{t} + \frac{1}{48}e^{3t}$

$= \frac{1}{2}e^{-2t} + \frac{3}{16}e^{-t} + \frac{1}{6} + \frac{1}{8}e^{t} + \frac{1}{48}e^{3t}$

K	-2	-1	0	1	2	3
$P_Z(K)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{6}$	$\frac{1}{8}$	0	$\frac{1}{48}$

b) $E(Z^3) = M^{(3)}(0) = \left(\frac{1}{2}(-2)^3e^{-2t} + \frac{3}{16}(-1)^3e^{-t} + \frac{1}{8}e^{t} + \frac{1}{48}3^3e^{3t}\right)\Big|_{t=0}$

$= \frac{-8}{2} - \frac{3}{16} + \frac{1}{8} + \frac{27}{48} = \frac{-7}{2}$

2. (8 points) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed Poisson random variables with parameter λ. Let

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(a) Compute the moment generating function of Y_n.

(b) Compute $E[Y_n^2]$.

a) $M_{Y_1}(t) = e^{\lambda(e^t - 1)}$

$$M_{Y_n}(t) = E\left[e^{\frac{t}{n}\sum_{i=1}^{n} X_i}\right] = E\left[e^{\frac{t}{n}X_1} \cdots e^{\frac{t}{n}X_n}\right] \overset{\text{independence}}{=} E\left[e^{\frac{t}{n}X_1}\right] \cdots E\left[e^{\frac{t}{n}X_n}\right]$$

$$= \left(E\left[e^{\frac{t}{n}X_1}\right]\right)^n = e^{n\lambda(e^{\frac{t}{n}} - 1)} = \exp\left(n\lambda e^{\frac{t}{n}} - 1\right)$$

b) $$E[Y_n^2] = M_{Y_n}^{(2)}(0) = \frac{d^2}{dt^2}\left(e^{n\lambda(e^{\frac{t}{n}}-1)}\right)\Big|_{t=0}$$

$$= \frac{d}{dt}\left(e^{n\lambda(e^{\frac{t}{n}}-1)} \frac{n\lambda e^{\frac{t}{n}}}{n}\right)\Big|_{t=0} = \frac{d}{dt}\left(\lambda e^{\frac{t}{n}} e^{n\lambda(e^{\frac{t}{n}}-1)}\right)\Big|_{t=0}$$

$$= \left(\frac{\lambda e^{\frac{t}{n}}}{n} e^{n\lambda(e^{\frac{t}{n}}-1)} + \lambda e^{\frac{t}{n}} e^{n\lambda(e^{\frac{t}{n}}-1)} \lambda e^{\frac{t}{n}}\right)\Big|_{t=0}$$

$$= \frac{\lambda}{n} + \lambda^2 = \lambda\left(\lambda + \frac{1}{n}\right)$$

3. **(6 points)** An ice cream shop estimates that the number of customers that arrive on a given day is a random variable with mean 1000 and variance 20^2. If more than 1500 customers purchase ice cream on a given day then the store runs out of ice cream. The owners of the shop do not know the distribution of X but they suspect that it is **not** normally distributed.

(a) Give the best possible upper bound on the probability that the store runs out of ice cream on Friday.

(b) If fewer than 400 customers arrive on a particular day than the ice cream shop loses money and must furlough staff. Give the best possible upper bound on the probability that the store does not furlough staff on Friday.

a) Chebyshev : $P(X \geq \mu + c) \leq \frac{\sigma^2}{c^2}$

$P(X \leq \mu - c) \leq \sigma^2/c^2$

$P(X \geq 1501) = P(X \geq 1000 + 501) \leq \frac{20^2}{501^2} = .001594$

b) $P(X \geq 400) \leq 1$

chebyshev gives an upper bound for $P(X \leq 399)$. However $P(X \geq 400) = 1 - P(X \leq 399)$ so Chebyshev would give a lower bound for $P(X \geq 400)$.

4. (10 points)

(a) Show that if $X \sim N(\mu, \sigma^2)$, then $P(X > 0) = \Phi(\frac{\mu}{\sigma})$ where Φ is the c.d.f. of the standard normal distribution.

$X \stackrel{d}{=} \sigma Z + \mu$ where $Z \sim N(0,1)$

$P(X>0) = P(\sigma Z + \mu > 0) = P\left(Z > -\frac{\mu}{\sigma}\right) = 1 - P\left(Z \le -\frac{\mu}{\sigma}\right)$

$= 1 - \Phi\left(-\frac{\mu}{\sigma}\right) = 1 - \left(1 - \Phi\left(\frac{\mu}{\sigma}\right)\right) = \Phi\left(\frac{\mu}{\sigma}\right)$

(b) Civil engineers are designing a bridge. For modeling purposes they assume that the weight (in the unit of 1000 pounds) of each car is an independent random variable with mean 3 and variance $(0.3)^2$. Suppose the amount of weight W the bridge can withstand without sustaining structural damage is a normal random variable with mean m and variance (10^2). How large should m be so that the probability of structural failure with 100 cars is less than 0.01?

Hint: Use the central limit theorem. Let $X_1, \ldots, X_n$ be independent and identically distributed with mean 3 and variance $(0.3)^2$. Structural damage corresponds to the event $\{S_n - W > 0\}$ where $S_n = X_1 + \cdots + X_n$.

$S_n - W$ has mean $3n - m$ and variance $.09n + 100$, $n = 100$

by CLT $S_n - W \approx N(3n - m, .09n + 100)$

$P(S_n - W > 0) < .01 \Rightarrow \Phi\left(\frac{3n-m}{\sqrt{.09n+100}}\right) < .01$

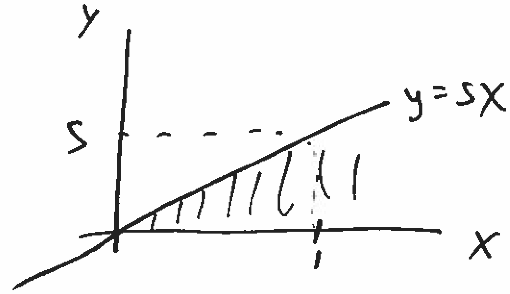
$\frac{3n-m}{\sqrt{.09n+100}} = -2.33 \Rightarrow m =$

$m = 324.29$ in units of 1000 lbs

5. (10 points) Let X and Y be jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} 6e^{-2x-3y}, & 0 < x < \infty,\ 0 < y < \infty \\ 0, & \text{otherwise.} \end{cases}$$

and let $Z = Y/X$.



(a) Find the cumulative distribution function of Z.

$$P(Z \le s) = P\left(\frac{Y}{X} \le s\right) = P(Y \le sX)$$

$$= \int_0^\infty \int_0^{sx} 6e^{-2x-3y}\,dy\,dx = \int_0^\infty 6e^{-2x} \int_0^{sx} e^{-3y}\,dy\,dx$$

$$= \int_0^\infty 6e^{-2x} \left(\frac{e^{-3y}}{-3}\right)\Bigg|_{y=0}^{sx} dx = \int_0^\infty -2e^{-2x}\left(e^{-3xs} - 1\right) = -2\int_0^\infty e^{-x(2+3s)} - e^{-2x}\,dx$$

$$= -2\left(\frac{e^{-x(2+3s)}}{-(2+3s)}\Bigg|_{x=0}^{\infty} - \frac{e^{-2x}}{-2}\Bigg|_{x=0}^{\infty} \right) = -2\left(\frac{1}{2+3s} - \frac{1}{2}\right) = 1 - \frac{2}{2+3s}$$

$$= \frac{3s}{2+3s}$$

$$F_Z(s) = \begin{cases} 0 & s < 0 \\ \dfrac{3s}{2+3s} & s \ge 0. \end{cases}$$

(b) Find the density function of Z.

$$f_Z(s) = F_Z'(s) = \frac{d}{ds}\left(\frac{3s}{2+3s}\right) = \frac{3(2+3s) - 3(3s)}{(2+3s)^2}$$

$$= \frac{6}{(2+3s)^2}$$

$$f_Z(s) = \begin{cases} 0 & s < 0 \\ \dfrac{6}{(2+3s)^2} & s \ge 0. \end{cases}$$

6. (10 points) Suppose we roll a fair, 6-sided die repeatedly. Let A_k be the indicator random variable for the event that the kth roll is a 1. Similarly, let B_k be the indicator random variable for the event that the kth roll is even.

(a) Find $E[A_iB_j]$ (consider both cases $i = j$ and $i \neq j$).

$i=j$: $E[A_iB_i] = 0$

$i \neq j$ $E[A_i B_j] = P(A_i=1, B_j=1) = \frac{1}{6} \times \frac{3}{6} = \frac{3}{36}$

(b) Out of n rolls of the die, let X be the number of 1's that occur and let Y be the number rolls that were even. Find $E[XY]$. Hint: part (a) may help.

$X = \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} A_i \quad Y = \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} B_i \quad \begin{pmatrix} X_i = A_i \\ Y_i = B_i \end{pmatrix}$

$E[XY] = E\left[\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} Y_i\right)\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} E[A_iB_j]$

$= (n^2-n)\frac{3}{36} = \boxed{\frac{n(n-1)}{12}}$

(c) Find $\mathrm{Cov}(X, Y)$.

$E[X] = \frac{n}{6} \qquad E[Y] = \frac{n}{2}$

$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{n(n-1)}{12} - \frac{n^2}{12}$

$= \boxed{\frac{-n}{12}}$