MTH 201

Midterm 1

Feb 27, 2020

Name: Key

UR ID: ____________________

Circle your Instructor's Name:

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Instructions:

• The presence of calculators, cell phones, and other electronic devices at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.

• For each problem, please put your final answer in the answer box. We will judge your work outside the box as well (unless specified otherwise) so you still need to show work and justify your answers. You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.

• In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$ and you can leave your answers in terms of $\binom{n}{k}$ or $k!$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Summations must also be evaluated, in particular, the symbols "$\sum$" or "$\cdots$" should not appear in final answers.

• This exam is out of 50 points. You are responsible for checking that this exam has all 10 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____________________

1. (4 points) For this problem, justification is not required and partial credit will **not** be awarded.

Suppose an urn contains ten chips labeled $1, \cdots, 10$. Five of the chips are black, three are red, and two are green. The chips are drawn randomly one at a time without replacement until the urn is empty.

(a) What is the probability that the 8th draw is the chip labeled 5?

$$\frac{9}{10} \quad \frac{8}{9} \quad \frac{7}{8} \quad \frac{6}{7} \quad \frac{5}{6} \quad \frac{4}{5} \quad \frac{3}{4} \quad \frac{1}{3} \quad \frac{2}{2} \quad \frac{1}{1}$$

Answer: $\frac{1}{10}$

(b) What is the probability that there is at least two black chips among the first 5 draws?

$P(\text{at least } 2) = 1 - P(0 \text{ or } 1)$

$$P(0) = \frac{\binom{5}{5}}{\binom{10}{5}} = \frac{1}{\binom{10}{5}}$$

$$P(1) = \frac{\binom{5}{4}\binom{5}{1}}{\binom{10}{5}} = \frac{25}{\binom{10}{5}}$$

Answer: $= 1 - \frac{1}{\binom{10}{5}}(1+25) = 1 - \frac{26}{\binom{10}{5}}$

2. (7 points) Alice, Bob, and Carmella are playing a dice game at the beautiful *Rochester Spa and Casino*. The host rolls a fair die. If a one is rolled, Alice wins \$1. If a two is rolled, Bob wins \$1. If a three is rolled, Carmella wins \$1. If a 4, 5, or 6 is rolled then nobody wins. The game is repeated 10 times. Find the probability that all players win at least \$1.

Hint: You might need the inclusion-exclusion formula

$$\mathbb{P}(\cup_{i=1}^{n} A_i) = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_k}).$$

A = ~~Alice wins~~ alice wins at least \$1

B = Bob wins at least \$1

C = Carmella wins at least \$1

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \wedge B) - P(B \wedge C) - P(A \wedge C) + P(A \wedge B \wedge C)$$

$$= 3P(A) - 2P(A \wedge B) + P(A \wedge B \wedge C)$$

$P(A) = 1 - P(\text{alice wins } 0) = 1 - \left(\frac{5}{6}\right)^{10}$

$P(A \wedge B) = 1 - P(\text{alice wins } 0, \text{Bob wins } 0) = 1 - \left(\frac{4}{6}\right)^{10}$

$P(A \wedge B \wedge C) = 1 - P(\text{alice wins } 0, \text{Bob wins } 0, \text{Carmella wins } 0) = 1 - \left(\frac{3}{6}\right)^{10}$

Answer:

$$P(A \cup B \cup C) = 3\left(1 - \left(\tfrac{5}{6}\right)^{10}\right) - 3\left(1 - \left(\tfrac{4}{6}\right)^{10}\right) + \left(1 - \left(\tfrac{3}{6}\right)^{10}\right) = 1 - 3\left(\tfrac{5}{6}\right)^{10} + 3\left(\tfrac{4}{6}\right)^{10} - \left(\tfrac{3}{6}\right)^{10}$$

$\approx .462$

3. (8 points) You are playing soccer (also called "football") next to a river. Each time you kick the ball there is 50% probability it goes into the goal and you score 1 point, 40% probability you miss the goal (worth 0 points) but do not lose the ball, and a 10% probability the ball falls into the river and is lost forever. You kick the ball repeatedly until it is lost.

(a) What is the probability the ball is lost eventually?

$P(\text{lost after } k \text{ kicks}) = (1-.1)^{k-1}(.1) = .9^{k-1} \cdot .1$

$P(\text{lost}) = \sum_{k=1}^{\infty} P(\text{lost after } k \text{ kicks}) = \sum_{k=1}^{\infty} (.1)(.9)^{k-1} = \frac{.1}{1-.9} = 1$

Answer: 1

(b) What is the probability you never score a point?

$P(\text{lost after } k \text{ kicks and never score}) = (1-.1-.5)^{k-1}(.1) = (.4)^{k-1}(.1)$

$P(\text{never score}) = \sum_{k=1}^{\infty} P(\text{lost after } k \text{ and never score}) = \sum_{k=1}^{\infty} (.4)^{k-1}(.1) = \frac{.1}{1-.4} = \frac{.1}{.6} = \frac{1}{6}$

Answer: $1/6$

4. (8 points) Let X be a random variable whose possible values lie in the interval $[0, 3]$ with probability density given by:

$$f(x) = \begin{cases} Cx^3, & 0 \le x \le 3 \\ 0, & \text{otherwise} \end{cases}$$

where C is a constant.

(a) (2pts) Find the constant C.

$$1 = \int f(x)\,dx = \int_0^3 Cx^3\,dx = C\frac{x^4}{4}\Big|_0^3 = \frac{C}{4}\,\cancel{\,}\,3^4 = \frac{81C}{4}$$

Answer:

$$C = \frac{4}{81}$$

(b) (1pt) What is the probability of the event "$X = 1$"?

$$\int_1^1 f(x)\,dx = 0$$

Answer:

0

(c) (2pts) What is the probability of the event "X lies between $\frac{1}{2}$ and 2"?

$$\int_{1/2}^{2} f(x)\,dx = \int_{1/2}^{2} cx^3\,dx = \frac{cx^4}{4}\Big|_{1/2}^{2}$$

$$= \frac{c}{4}\left(2^4 - 2^{-4}\right)$$

$$= \frac{1}{81}\left(16 - \frac{1}{16}\right)$$

Answer:

$$\frac{1}{81}\left(16 - \frac{1}{16}\right)$$

(d) (3pts) Find the cumulative distribution function $F(x)$ for the variable X.

$$\cancel{F(s) = \int_0^X cx^3\,dx =}$$

$$F(s) = \int_0^s cx^3\,dx = \frac{cs^4}{4} = \frac{1}{81}s^4$$

$$F(s) = \begin{cases} 0 & s \le 0 \\ s^4/81 & 0 \le s \le 3 \\ 1 & s > 3. \end{cases}$$

Answer:

5. (8 points) Suppose $P(A) = 1/5$, $P(B) = 2/5$, and $P(A \cup B) = 2/5$.

(a) Are A and B independent?

$P(A \cup B) = P(A) + P(B) - P(A \wedge B)$

$P(A \wedge B) = P(A) + P(B) - P(A \cup B) = 1/5$

$P(A)\,P(B) = 2/5 \neq P(A \cap B)$

Answer:

NO

(b) Are A and B conditionally independent given the event $A \cup B$?

$\cancel{P(A|B) = \frac{P(A \wedge B)}{P(B)} = \frac{1/5}{2/5} = 1/2}$

$P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{1/5}{2/5} = 1/2$

$P(B|A \cup B) = \frac{P(B \wedge (A \cup B))}{P(A \cup B)} = \frac{P(B)}{P(A \cup B)} = \frac{2/5}{2/5} = 1$

$P(A \cap B|A \cup B) = \frac{P((A \wedge B) \wedge (A \cup B))}{P(A \cup B)} = \frac{P(A \wedge B)}{P(A \cup B)} = \frac{1/5}{2/5} = 1/2$

$P(A|A \cup B)\,P(B|A \cup B) = P(A \cap B\,|\,A \cup B)$

Answer:

yes

6. (8 points)

(a) A family has 2 children. What is the probability that both are boys, given that at least one is a boy? You may assume the gender of each child is independent of the gender of the other children and that boys and girls occur with equal frequency.

B G

G B

B B

Answer: $1/3$

(b) A family has n children, at least $n-1$ of which are boys. What is the probability that all n children are boys?

$$P(n \text{ boys} \mid \overset{\text{at least}}{n-1} \text{ boys}) = \frac{P(n \text{ boys})}{P(\text{at least } n-1 \text{ boys})} = \frac{(1/2)^n}{n(\frac{1}{2})^n + (1/2)^n} = \frac{1}{1+n}$$

$$P(n \text{ boys}) = \left(\tfrac{1}{2}\right)^n$$

$$P(n-1 \text{ boys}) = \binom{n}{1}\left(\tfrac{1}{2}\right)^n = n\left(\tfrac{1}{2}\right)^n$$

Answer: $\frac{1}{1+n}$

7. (7 points) There are 2 factories which manufacture silicon chips. 2/3 of the chips from factory I are defective while 1/10 of the chips from factory II are defective. Factory I produces 3 times as many chips as factory II each week. A chip is chosen at random and found to not be defective. What is the probability that this chip came from factory II?

$$3P(\text{factory II}) = P(\text{factory I})$$

$$P(I) + P(II) = 1$$

$$3P(II) + P(II) = 1 \implies P(II) = 1/4 \quad P(I) = 3/4$$

$$P(II \mid \text{not defective}) = \frac{P(\text{not defective} \mid II)\,P(II)}{P(\text{not defective} \mid I)\,P(I) + P(\text{not defective} \mid II)\,P(II)}$$

$$= \frac{\left(\frac{9}{10}\right)\left(\frac{1}{4}\right)}{\left(\frac{1}{3}\right)\left(\frac{3}{4}\right) + \left(\frac{9}{10}\right)\left(\frac{1}{4}\right)} = \frac{9/10}{1 + 9/10} = \frac{9/10}{19/10}$$

$$= \frac{9}{19}$$

Answer: 9/19

EXTRA PAGE. You may use this page if you run out of space. Be sure to label your problems on this page and also include a note on the original page telling the graders to look for your work here.