AM exam- Solutions

1. (10 points) Factories in Albany and Buffalo both produce the same type of gadget. On average 15% of the gadgets produced by Albany are mildly defective (they use cheap parts) and 3% of the gadgets produced by Buffalo are mildly defective. The NY State inspectors visit each factory and take a random sample of 100 gadgets. If more than 8 gadgets from the sample are mildly defective, they will shutdown production. Let A be the event that the Albany factory gets shutdown and let B be the event that the Buffalo factory gets shutdown.

$X = \#$ Albany defectives | $Y = \#$ for Buffalo

(a) Give an exact expression for $P(A)$, without attempting to evaluate it.

$$P(A) = P(X>8) = \sum_{k=9}^{100} \binom{100}{k} (0.15)^k (0.85)^{100-k}$$

(b) Determine whether the normal or the Poisson approximation is appropriate for approximating $P(A)$, justifying your answer.

$np(1-p) = 100(0.15)(0.85) = 12.75 > 10$ norm is good.

$np^2 = 100(0.15)^2 = 2.25 \to$ not small Poisson bad

(c) Use either the normal or the Poisson approximation, whichever is appropriate, to give an approximation of $P(A)$.

$$P(A) = P(X>8) = \cancel{P\left(\frac{X-15}{\sqrt{12.75}}\right)}$$

$$= P(X>8.5) = P\left(\frac{X-15}{\sqrt{12.75}} > \frac{-6.5}{\sqrt{12.75}}\right)$$

$Z \sim N(0,1) \longrightarrow \approx P\left(Z > \frac{-6.5}{\sqrt{12.75}}\right) \approx 1 - 0.03435$

(d) Determine whether the normal or the Poisson approximation is appropriate for approximating $P(B)$, justifying your answer.

$np(1-p) = 2.91 < 10 \to$ norm bad

$np^2 = 100(0.03)^2 = 0.09 \to$ Poisson good

(e) Use either the normal or the Poisson approximation, whichever is appropriate, to give an approximation of $P(B)$.

$$P(B) = P(Y>8) = 1 - P(Y \leq 8)$$

$$= 1 - \sum_{k=0}^{8} e^{-3} \frac{3^k}{k!} \approx 1 - 0.996 = \boxed{0.0038}$$

2. (6 points) WeLovePolls.com recently conducted a poll of $n = 1000$ likely voters asking whether they intend to vote for candidates Boc Jiden or Sernie Banders in the upcoming election. 53% said they will vote for Jiden, 47% said they will vote for Banders. WeLovePolls.com reports that the confidence interval for p, the true population proportion of voters that intend to vote for Jiden on election day, is $(0.50, 0.56)$. What is the confidence level for the reported confidence interval?

$$P(|p - 0.53| < 0.03) \geq 2\Phi(2\epsilon\sqrt{n}) - 1$$

$$= 2\Phi(2(0.03)\sqrt{1000}) - 1$$

$\epsilon = 0.03$

$n = 1000$

$$\approx 2\Phi(1.897367) - 1$$

$$= 0.9422$$

94.22%

3 **(9 points)** Let $Z \sim \mathcal{N}(0,1)$ and $X \sim \mathcal{N}(\mu, \sigma^2)$. This means that Z is a standard normal random variable with mean 0 and variance 1, while X is a normal random variable with mean μ and variance σ^2. Note: for each part, you will not earn points if you do not use the method indicated.

(a) Use integration by parts to show the reduction formula:

Can use L'Hospital's rule $\lim \frac{x^{n-1}}{e^{x^2/2}} \frac{0}{0} = \lim \frac{(n-1)x^{n-2}}{x e^{x^2/2}} = \lim \frac{(n-1)x^{n-3}}{e^{x^2/2}} = \cdots \to 0$ need x

$$\int_{-\infty}^{\infty} x^n e^{-x^2/2}\, dx \;=\; \int_{-\infty}^{\infty} (n-1)x^{n-2}e^{-x^2/2}\, dx \quad \text{for} \quad n \geq 2.$$

$u = x^{n-1}$

$du = (n-1)x^{n-2}dx \;\Rightarrow\; \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx = \cancel{-x^{n-1} e^{-x^2/2}\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty} (n-1)x^{n-2} e^{-x^2/2} dx$

$dv = x e^{-x^2/2} dx$

$v = -e^{-x^2/2}$

$= \int_{-\infty}^{\infty} (n-1)x^{n-2} e^{-x^2/2} dx$

(b) Use the formula from part (a) to show $E(Z^4) = 3$ and $E(Z^3) = 0$.

$Z \sim \mathcal{N}(0,1) \Rightarrow E(Z) = 0$ and $1 = Var(Z) = E(Z^2) - \underbrace{E(Z)}_{0}$

$\Rightarrow E(Z^2) = 1$

From (a), $E(Z^4) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^4 e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 3x^2 e^{-x^2/2} dx = 3E(Z^2) = 3$

$E(Z^3) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^3 e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} 2x e^{-x^2/2} = 3E(Z) = 0.$

(c) Use the result in part (b) to show $E(X^4) = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$.

$X = \sigma Z + \mu \Rightarrow X^4 = (\sigma Z + \mu)^4 = (\sigma Z)^4 + 4(\sigma Z)^3 \mu + 6(\sigma Z)^2 \mu^2 + 4\sigma Z \mu^3 + \mu^4$

$\Rightarrow E(X^4) = \sigma^4 E(Z^4) + 4\sigma^3 \mu \underbrace{E(Z^3)}_{0} + 6\sigma^2\mu^2 \underbrace{E(Z^2)}_{1} + 4\sigma\mu^3 \underbrace{E(Z)}_{0} + \mu^4$

$= 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$.

4. (8 points) Let $X \sim \mathrm{Exp}(\lambda)$. $\Rightarrow E(X) = \frac{1}{\lambda}$, $Var(X) = \frac{1}{\lambda^2}$

(a) Compute $E[3X + 5]$.

$$= 3E(X) + 5$$

$$= \frac{3}{\lambda} + 5$$

(b) Compute $\mathrm{Var}[3X + 5]$. $= 3^2 Var(X)$

$$= \frac{9}{\lambda^2}$$

5. (9 points) Let X be a random variable with CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^2 & \text{if } 0 \le x < 1/2, \\ x/2 & \text{if } 1/2 \le x < 2 \\ 1 & \text{if } \cancel{1 \le x}, \; 2 \le X \end{cases}$$

(a) Compute the PDF of X.

$$f(x) = F'(x) = \begin{cases} 0 & , \; X < 0 \\ 2X & , \; 0 \le X < 1/2 \\ \frac{1}{2} & , \; 1/2 \le X < 2 \\ 0 & , \; 2 \le X \end{cases}$$

(b) Compute $E[X]$. $= \int x f(x) dx$

$$= \int_0^{1/2} 2x^2 dx + \int_{1/2}^{2} \frac{x}{2} dx$$

$$= \frac{2}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{4}\left(2^2 - \left(\frac{1}{2}\right)^2\right) = \frac{49}{48}$$

(c) Compute $\text{Var}(X)$.

$$E(x^2) = \int_0^{1/2} 2x^3 dx + \int_{1/2}^{2} \frac{x^2}{2} dx = \frac{1}{2}\left(\frac{1}{2}\right)^4 + \frac{1}{6}\left(2^3 - \left(\frac{1}{2}\right)^3\right)$$

$$= \frac{1}{32} + \frac{42}{32} = \frac{43}{32}$$

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{43}{32} - \left(\frac{49}{48}\right)^2 = \boxed{0.301649}$$

6. (8 points) You are tasked with operating a machine in a factory which malfunctions every X hours where X is a random variable with exponential distribution. You are told that on average the machine malfunctions every 6 hours. Your shift lasts 8 hours. If the machine malfunctions you are sent home early and paid \$20 for your day of work. If the machine does not malfunction you earn \$100 for your day of work. If the machine is functioning when you arrive how much do you expect to earn for the day?

$$X \sim Exp(\tfrac{1}{6}) \Rightarrow E(X) = 6 \text{ hrs.}$$

$$F(X) = \text{pay} = \begin{cases} 20, & \text{if } X < 8 \\ 100, & \text{if } X > 8 \end{cases}$$

$$E(F(X)) = 20\,P(X<8) + 100\,P(X>8)$$

$$= 20\int_0^8 \tfrac{1}{6} e^{-x/6}\,dx + 100\int_8^\infty \tfrac{1}{6} e^{-x/6}\,dx$$

$$= -20\,(e^{-4/3} - 1) + 100\left(-e^{-x/6}\right)\Big|_8^\infty$$

$$= 20 - 20e^{-4/3} + 100\,e^{-4/3}$$

$$= 20 + 80e^{-4/3}$$