PM exam - Solutions

1. (10 points) Anne and Bob both enjoy swimming. Every day they each decide whether or not to go swimming independently from previous days and independently from each other. On average Anne goes swimming 1% of days and Bob goes swimming 20% of days. Let A be the event that Anne goes swimming less than 7 days in the next year. Let B be the event that Bob goes swimming less than 65 days in the next year. Assume there are 365 days in a year. $X = \#$ Anne swim days | $Y = \#$ days Bob.

(a) Give an exact expression for $P(A)$, without attempting to evaluate it.

$$P(A) = P(X<7) = \sum_{k=0}^{6} \binom{365}{k} (0.01)^k (0.99)^{365-k}$$

(b) Determine whether the normal or the Poisson approximation is appropriate for approximating $P(A)$, justifying your answer.

$$np(1-p) = 365(0.01)(0.99) = 3.6135 < 10$$ norm bad

$$np^2 = 365(0.01)^2 = 0.0365 \to$$ Poisson good.

(c) Use either the normal or the Poisson approximation, whichever is appropriate, to give an approximation of $P(A)$.

$$P(A) = P(X<7) = \sum_{k=0}^{6} e^{-3.65} \frac{3.65^k}{k!} \approx 0.9225$$

Use $X \sim$ Poisson(3.65)

(d) Determine whether the normal or the Poisson approximation is appropriate for approximating $P(B)$, justifying your answer.

$$np(1-p) = 58.4 \to$$ norm. good

$$np^2 = 14.6 \to$$ Poisson bad

(e) Use either the normal or the Poisson approximation, whichever is appropriate, to give an approximation of $P(B)$.

$$P(B) = P(Y<65) = P(Y<64.5)$$

$$= P\left(\frac{Y-73}{\sqrt{58.4}} < \frac{-8.5}{\sqrt{58.4}}\right)$$

$Z \sim N(0,1)$

$$\approx P\left(Z < \frac{-8.5}{\sqrt{58.4}}\right) \approx 0.13301$$

2. (8 points) You are tasked with operating a machine in a factory which malfunctions every X hours where X is a random variable with exponential distribution. You are told that on average the machine malfunctions every 8 hours. Your shift lasts 8 hours. If the machine malfunctions you are sent home early and paid \$10 for your day of work. If the machine does not malfunction you earn \$100 for your day of work. If the machine has been functioning for 8 hours when you arrive how much do you expect to earn for the day?

$\hookrightarrow X \sim \text{Exp}(\frac{1}{8})$ is memoryless

$E(X) = 8$

$$F(X) = \text{pay} = \begin{cases} 10, & \text{if } X < 8 \\ 100, & \text{if } X > 8 \end{cases}$$

$$E(F(X)) = 10\,P(X<8) + 100\,P(X>8)$$

$$= 10\int_0^8 \frac{1}{8} e^{-x/8}\,dx + 100\int_8^\infty \frac{1}{8} e^{-x/8}\,dx$$

$$= 10\left(1 - \frac{1}{e}\right) + 100\left(-e^{-x/8}\right)\Big|_8^\infty$$

$$= 10 + \frac{90}{e}$$

3. (9 points) Let X be a random variable with CDF

$$F(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^3 & \text{if } 0 \le x < 1/2, \\ x/4 & \text{if } 1/2 \le x < 4 \\ 1 & \text{if } 4 \le x, \end{cases}$$

(a) Compute the PDF of X.

$$f(x) = F'(x) = \begin{cases} 0, & x < 0 \\ 3x^2, & 0 \le x < 1/2 \\ 1/4, & \frac{1}{2} \le x < 4 \\ 0, & x \ge 4 \end{cases}$$

(b) Compute $E[X]$.

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{1/2} 3x^3\,dx + \int_{1/2}^{4} \frac{x}{4}\,dx$$

$$= \frac{3}{4}\left(\frac{1}{2}\right)^4 + \frac{1}{8}\left(4^2 - \left(\frac{1}{2}\right)^2\right)$$

$$= \frac{129}{64}$$

(c) Compute $\mathrm{Var}(X)$.

$$E(X^2) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{1/2} 3x^4\,dx + \int_{1/2}^{4} \frac{x^2}{4}\,dx$$

$$= \frac{3}{5}\left(\frac{1}{2}\right)^5 + \frac{1}{12}\left(4^3 - \left(\frac{1}{2}\right)^3\right)$$

$$= \frac{2564}{480}$$

$$\Rightarrow \mathrm{Var}(X) = E(X^2) - \mu^2$$

$$= \frac{2564}{480} - \left(\frac{129}{64}\right)^2 \approx 1.279$$

4. (8 points) Let $X \sim \text{Bin}(n, p)$.

(a) Compute $E[2X + 3]$.

$= 2E(X) + 3$

$= 2np + 3$

(b) Compute $\text{Var}[2X + 3]$. $= 2^2 \text{Var}(X)$

$= 4np(1-p)$

5 (9 points) Let $Z \sim \mathcal{N}(0,1)$ and $X \sim \mathcal{N}(\mu, \sigma^2)$. This means that Z is a standard normal random variable with mean 0 and variance 1, while X is a normal random variable with mean μ and variance σ^2. Note: for each part, you will not earn points if you do not use the method indicated.

(a) Use integration by parts to show the reduction formula:

$$\int_{-\infty}^{\infty} x^n e^{-x^2/2}\, dx = \int_{-\infty}^{\infty} (n-1)x^{n-2} e^{-x^2/2}\, dx \quad \text{for} \quad n \geq 2.$$

Can use L'Hospital's rule $\to \lim \frac{x^{n-1}}{e^{x^2/2}} \frac{\infty}{\infty} = \lim \frac{(n-1)x^{n-2}}{x e^{x^2/2}} = \lim \frac{(n-1)x^{n-3}}{e^{x^2/2}} = \cdots \to 0$ repeat

$u = x^{n-1}$

$du = (n-1)x^{n-2}dx$

$dv = xe^{-x^2/2}dx$

$v = -e^{-x^2/2}$

$\to \int_{-\infty}^{\infty} x^n e^{-x^2/2} dx = \cancel{-x^{n-1}e^{-x^2/2}\Big|_{-\infty}^{\infty}} + \int_{-\infty}^{\infty}(n-1)x^{n-2}e^{-x^2/2}dx$

$= \int_{-\infty}^{\infty}(n-1)x^{n-2}e^{-x^2/2}dx$

(b) Use the formula from part (a) to show $E(Z^4) = 3$ and $E(Z^3) = 0$.

$Z \sim \mathcal{N}(0,1) \Rightarrow E(Z) = 0$ and $1 = Var(Z) = E(Z^2) - \underbrace{E(Z)}_{0}$

$\Rightarrow E(Z^2) = 1$

From (a), $E(Z^4) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^4 e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 3x^2 e^{-x^2/2}dx = 3E(Z^2) = 3$

$E(Z^3) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^3 e^{-x^2/2}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} 2xe^{-x^2/2} = 3E(Z) = 0.$

(c) Use the result in part (b) to show $E(X^4) = 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4$.

$X = \sigma Z + \mu \Rightarrow X^4 = (\sigma Z + \mu)^4 = (\sigma Z)^4 + 4(\sigma Z)^3\mu + 6(\sigma Z)^2\mu^2 + 4\sigma Z\mu^3 + \mu^4$

$\Rightarrow E(X^4) = \sigma^4 E(Z^4) + 4\sigma^3\mu \underbrace{E(Z^3)}_{0} + 6\sigma^2\mu^2 \underbrace{E(Z^2)}_{1} + 4\sigma\mu^3 \underbrace{E(Z)}_{0} + \mu^4$

$= 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4.$

6. (6 points) An important engine part is built and shipped from the Buffalo Truck Parts Factory. Before a large shipment goes out, the quality control manager Alice needs to estimate p, the true proportion of parts in the shipment that are mildly defective. Alice takes a sample of $n = 400$ parts and finds that 5% are mildly defective. She then tells her boss that she has 80% confidence that p lies in a confidence interval (a, b), but she has forgotten what a and b are. What is the interval (a, b)?

$$P(|p - 0.05| < \epsilon) \geq 0.8$$

$\hookrightarrow$ Use

$$2\Phi(2\epsilon\sqrt{n}) - 1 \geq 0.8$$

$$\Phi(2\epsilon\sqrt{n}) \geq 0.9$$

$n = 400 \longrightarrow$ $\Phi(40\epsilon) \geq 0.9$

$$40\epsilon = 1.29$$

$$\epsilon = 0.03225$$

$\rightarrow$ $(a, b) = (0.05 - \epsilon,\ 0.05 + \epsilon) = (0.01775,\ 0.08225)$