MTH 201

Final Exam

May 10, 2021

Name: Key

UR ID: ____________________

Circle your Instructor's Name:

Ian Alevy Thomas Tucker

Instructions:

- The presence of calculators, cell phones, and other electronic devices (other than a device for being on Zoom during the exam and uploading your exam afterwards) at this exam is strictly forbidden. Notes or texts of any kind are strictly forbidden.
- In your answers, you do not need to simplify arithmetic expressions like $\sqrt{5^2 - 4^2}$ and you can leave your answers in terms of $\binom{n}{k}$ or $k!$. However, known values of functions should be evaluated, for example, $\ln e, \sin \pi, e^0$. Summations must also be evaluated, in particular, the symbols "$\sum$" or "$\cdots$" should not appear in final answers.
- You must complete both parts A and B. Each part is out of 50 points. You are responsible for checking that this exam has all 14 pages.

PLEASE COPY THE HONOR PLEDGE AND SIGN:

I affirm that I will not give or receive any unauthorized help on this exam, and all work will be my own.

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YOUR SIGNATURE:____________________

Part A

1. (12 points) Dr. Prob has on his desk a biased coin with $P(H) = 1/4$ and $P(T) = 3/4$. Each day he flips the coin 80 times and records the sequence of flips.

The last page of the exam has a table with the CDF of the normal distribution. Use this table to give numerical quantities for any expressions involving $\Phi(x)$.

(a) Approximate the probability that he records 60 or fewer tails using either the normal or the Poisson approximation, whichever is more appropriate (and justify your choice).

$np(1-p) = 80 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{60}{4} = 15 \Rightarrow$ normal approx.

$np^2 = \frac{80}{16}$. $X_n = \#$ of tails in n flips

$P(X_n \leq 60) = P\left(\frac{X_n - E(X_n)}{Var(X_n)} \leq \frac{60-60}{Var(X_n)} \right) \approx P(Z \leq 0) = \frac{1}{2}$

(b) If Dr. Prob observes fewer than 60 tails each day in a given week then he buys a lottery ticket. Over the course of 10 weeks, approximate the probability that he buys at most 2 lottery tickets using either the normal or Poisson approximation, whichever is more appropriate (and justify your choice).

$Y =$ # of lottery tickets

$P(\text{buys a ticket in a week}) = \left(\frac{1}{2}\right)^7$

$np^2 = 10 \cdot \left(\left(\frac{1}{2}\right)^7\right)^2 \rightarrow$ small $\rightsquigarrow$ Poisson approx

$Y \simeq \text{Poisson}\left(10\left(\frac{1}{2^7}\right)\right)$

$P(Y \leq 2) = e^{-\frac{10}{2^7}} + \frac{10}{2^7} e^{-\frac{10}{2^7}} + \left(\frac{10}{2^7}\right)^2 \frac{1}{2} e^{-\frac{10}{2^7}} = e^{-\frac{10}{2^7}}\left(1 + \frac{10}{2^7} + \left(\frac{10}{2^7}\right)^2 \frac{1}{2}\right)$

$= \frac{8857}{8192} e^{-\frac{10}{2^7}} = \frac{8857}{8192} e^{-5/64}$

2. (8 points) Let $X \sim \text{Bern}(1/5)$ and $G \sim \text{N}(0,1)$. Assume that X and G are independent. We are conducting an experiment in which we would like to measure X but due to the presence of noise in the signal can only measure $O = X + G$. Find the conditional probability

$$P(X = 1 | O > 1).$$

Express your answer as a fraction using the approximation $\Phi(1) \approx 21/25$ where $\Phi(x)$ is the CDF of the normal distribution.

$$P(X=1|O>1) = \frac{P(X=1, O>1)}{P(O>1)} = \frac{P(O>1 / X=1)\, P(X=1)}{P(O>1)}$$

$$= \frac{P(G>0)\ 1/5}{P(O>1, X=1) + P(O>1, X=0)} = \frac{(1/2)(1/5)}{(\frac{1}{2})(\frac{1}{5}) + (1-\frac{21}{25})\, 4/5}$$

$$= \frac{1/2}{1/2 + \frac{4}{25} \cdot 4} = \frac{1}{1+\frac{32}{25}} = \frac{25}{57}$$

3. **(12 points)** In each of the following questions, the random variable described is either Bernoulli, binomial, geometric, exponential, Poisson, or normal. For each question identify the distribution and give reasonable numerical values for all of its parameters. In some cases more than one random variable type might fit, but make sure that you select a choice where you can describe numerical values for all the parameters. No explanation is necessary for this problem. Only give your answers.

(a) Ian is shooting basketballs into a hoop. Assume that on each shot the probability that Ian makes the shot is 2/3 and that all shots are independent. Suppose Ian shoots the ball 50 times and let X_1 be the number of shots he makes. What type of random variable is X_1 and what are its parameters?

$$X_1 \sim \mathrm{Bin}(50, 2/3)$$

(b) An urn contains 10 balls, 4 of which are red and 6 green. You sample 4 balls uniformly without replacement. If you choose a green ball you win 1 dollar, otherwise you win nothing. Let X_2 be the amount of money you win. What type of random variable is X_2 and what are its parameters?

$$X_2 \sim \mathrm{Ber}\left(1 - \frac{1}{\binom{10}{4}}\right) = \mathrm{Ber}\left(1 - \frac{4!\,6!}{10!}\right)$$
$$= \mathrm{Ber}\left(1 - \frac{24}{5040}\right) = \mathrm{Ber}\left(\frac{209}{210}\right)$$

(c) Aircraft pilots occasionally observe *unidentified flying objects* or UFOs. Suppose that on average, a pilot will see 1 UFO per 5,000 hours of flight time. Suppose that a pilot is taking his first flight today and let X_3 be the amount of time, measured in hours, until he sees his first UFO. Assuming X_3 is a continuous random variable, what is the most convenient choice for the type of random variable and what are its parameters?

$$X_3 \sim \mathrm{Exp}\left(\frac{1}{5000}\right)$$

(d) Referring to part (c), let X_4 be the number of UFOs that a pilot sees in 500 hours of flight time. What type of random variable is X_4 and what are its parameters?

$$X_4 \sim \mathrm{Poisson}\left(\frac{500}{5000}\right) = \mathrm{Poisson}\left(\frac{1}{10}\right)$$

4. (10 points) A game is played as follows. A fair die is rolled repeatedly. If a 2 or 3 appears, then Bob gets a point, and the game continues. If a 4,5, or 6 appears, then Bob does not get a point but the game does continue. If a 1 appears the game ends.

(a) Find the expected number of rolls before a 1 appears and the game ends (for example, if a 1 first appeared on the fourth role, the number of roles before the game ends is 4).

X_a = # of rolls before a 1 appears

$X \sim \text{Geom}(1/6)$

$$E[X] = \frac{1}{1/6} = 6.$$

(b) Find the chance that Bob gets at least one point before the game ends.

~~$P(\text{at least one point}) = \sum_{k=1}^{\infty} P(\text{at least one point}, X = k) =$~~

$$P(\text{at least one point}) = 1 - P(\text{no points}) = 1 - \sum_{k=1}^{\infty} P(\text{no points}, X = k) = 1 - \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k-1} \frac{1}{6}$$

$$= 1 - \frac{1}{6}\left(\frac{1}{1 - 1/2}\right) = 1 - \frac{1}{3} = 2/3$$

5. (8 points) An urn contains 1 green ball, 1 red ball, and 1 yellow ball. I draw 4 balls with replacement. What is the probability that there is at least one color that is repeated exactly twice?

$$P(\text{all same color}) = \frac{3}{3^4} = \frac{1}{27}$$

$$P(\text{one color repeated 3 times}) = 3\left(\frac{1\times 2}{3^4}\right) = \frac{2}{27}$$

$$P\left(\begin{array}{c}\text{one color repeated}\\ \text{exactly twice}\end{array}\right) = 1 - \frac{3}{27} = \frac{24}{27} = \frac{8}{9}$$

$$P(2R \cup 2G \cup 2Y) = P(2R) + P(2G) + P(2Y) - P(2R, 2G) - P(2R, 2Y)$$
$$- P(2Y, 2G) + P(2R, 2G, 2Y)$$

$$= \frac{\binom{4}{2}\times 2\times 2}{3^4}\times 3 - 3\times\left(\frac{\binom{4}{2}}{3^4}\right) + 0$$

$$= \frac{24}{27} - \frac{6}{27} = \frac{18}{27} = \boxed{\frac{2}{3}}$$

Part B

1. (6 points) Suppose X and Y are independent random variables with moment generating functions

$$M_X(t) = \frac{1}{4}e^{-t} + \frac{1}{4} + \frac{1}{2}e^{t}$$
$$M_Y(t) = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}.$$

Let $Z = X + Y$.

(a) Compute the p.m.f. of Z.

$$M_{X+Y}(t) = \left(\frac{1}{4}e^{-t} + \frac{1}{4} + \frac{1}{2}e^{t}\right)\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}\right)$$

$$= \frac{1}{6}e^{-2t} + \frac{1}{12}e^{t} + \frac{1}{6}e^{-t} + \frac{1}{12}e^{2t} + \frac{1}{3} + \frac{1}{6}e^{3t}$$

$$= \frac{1}{6}e^{-2t} + \frac{1}{6}e^{-t} + \frac{1}{3} + \frac{1}{12}e^{t} + \frac{1}{12}e^{2t} + \frac{1}{6}e^{3t}$$

$P(Z=-2) = \frac{1}{6}$ $\quad P(Z=1) = 1/12$

$P(Z=-1) = \frac{1}{6}$ $\quad P(Z=2) = 1/12$ $\quad P(Z=k)=0$ for all other k

$P(Z=0) = \frac{1}{3}$ $\quad P(Z=3) = 1/6$

(b) Compute $E[Z^3]$.

$E[Z^3] = M^{(3)}(0)$. $\quad M^3(0) = \frac{1}{6}(-2)^3 + \frac{1}{6}(-1)^3 + \frac{1}{12} + \frac{1}{12}2^3 + \frac{1}{6}3^3$

$$= -\frac{4}{3} - \frac{1}{6} + \frac{1}{12} + \frac{2}{3} + \frac{27}{6} = \frac{15}{4}$$

2. (12 points) Let X and Y be jointly continuous random variables with joint density

$$f_{X,Y}(x,y) = \begin{cases} xe^{-x(1+y)}, & x > 0 \text{ and } y > 0 \\ 0, & \text{else} \end{cases}$$

(a) Find the marginal probability density function f_X for X.

$$f_X(x) = \int_0^\infty f_{xy}(x,y)\,dy = \int_0^\infty x e^{-x(1+y)}\,dy = xe^{-x}\int_0^\infty e^{-xy}\,dy$$

$$= \frac{xe^{-x}}{-x}\, e^{-xy}\Big|_{y=0}^{\infty} = e^{-x}$$

$$f_X(x) = \begin{cases} 0 & x \leq 0 \\ e^{-x} & x > 0. \end{cases}$$

(b) Let $Z = XY$. Find the cumulative distribution function of Z.

$$P(XY \leq t) = \int_0^\infty \int_0^{t/x} xe^{-x(1+y)}\,dy\,dx = \int_0^\infty -e^{-x}\left(e^{-xy}\right)\Big|_{y=0}^{t/x}\,dx$$

$$= \int_0^\infty -e^{-x}(e^{-t} - 1)\,dx = (1 - e^{-t})$$

$$P(Z \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 - e^{-t} & \text{if } t \geq 0. \end{cases}$$

(c) As in (b), let $Z = XY$. Find the probability density function of Z.

$$\frac{d}{dt} P(Z \leq t) = \begin{cases} 0 & \text{if } t < 0 \\ e^{-t} & \text{if } t \geq 0. \end{cases}$$

3. (10 points) An urn is filled with balls labeled $1, 2, 3 \ldots, 10$. Balls $1-7$ are colored blue while balls $8-10$ are colored red. Balls are sampled repeatedly from the urn with replacement. Let A_k be the indicator random variable for the event that the k-th draw is odd and B_k be the indicator random variable for the event that the k-th draw is colored red.

(a) Find $E[A_i B_j]$ (consider both cases $i = j$ and $i \neq j$).

$i \neq j$: $E[A_i B_j] = E[A_i]E[B_j] = \frac{1}{2} \cdot \frac{3}{10} = \frac{3}{20}$

$i = j$: $E[A_i B_i] = \frac{1}{10}$

(b) Out of n draws from the urn, let X be the number of draws where an odd numbered ball is drawn and let Y be the number of draws where a red ball is drawn. Find $E[XY]$. Hint: part (a) may help.

$X = \sum_{i=1}^{n} A_i \qquad Y = \sum_{i=1}^{n} B_i$

$E(XY) = E\left(\sum_{i=1}^{n} A_i \sum_{i=1}^{n} B_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} E(A_i B_j)$

$= (n^2 - n) E[A_1 B_2] + nE[A_1 B_1]$

$= (n^2 - n)\frac{3}{20} + n \cdot \frac{1}{10} = \frac{3}{20}n^2 - \frac{n}{20} = \frac{n}{20}(3n - 1)$

(c) Find $\text{Cov}(X, Y)$.

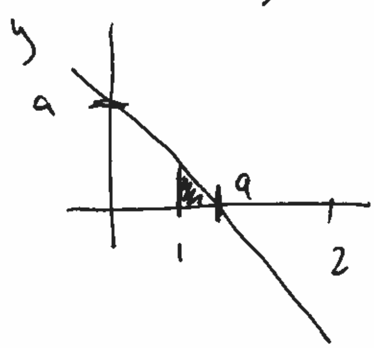
$\text{Cov}(X,Y) = E[XY] - E[X]E[Y] = \frac{n}{20}(3n-1) - \frac{3n^2}{20} = \frac{-n}{20}$

$E[X] = \sum_{i=1}^{n} E[A_i] = n \cdot \frac{1}{2} = \frac{n}{2}$

$E[Y] = \sum_{i=1}^{n} E[B_i] = n \cdot \frac{3}{10} = \frac{3n}{10}$

4. (6 points) Let X be uniformly distributed on $[1, 2]$ and let $Y \sim \text{Exp}(3)$. Assume X, Y are independent. Find $P(X + Y \leq a)$ for $1 \leq a \leq 2$. (Your answer should be a function of a).

$$f_X(x) = \begin{cases} 1 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{else} \end{cases} \qquad f_Y(y) = \begin{cases} 3e^{-3y} & \text{if } y > 0 \\ 0 & \text{else} \end{cases}$$



$$Y \leq a - X$$

$$P(Y + X \leq a) = \int_1^a \int_0^{a-x} 3e^{-3y}\,dy\,dx =$$

$$\int_1^a \left(1 - e^{-3(a-x)}\right) dx = a - 1 - \int_1^a e^{-3a+3x}\,dx$$

$$= a - 1 - e^{-3a} \int_1^a e^{3x}\,dx = a - 1 - \frac{e^{-3a}}{3}\left(e^{3a} - e^{3}\right)$$

$$= a - 1 - \frac{1}{3} + \frac{e^{-3a+3}}{3} = a - \frac{4}{3} + \frac{e^{3(1-a)}}{3}$$

5. (6 points) A fair coin is flipped ten times. Find the expected value for the number of times you see three tails in a roll. (For example, if you the sequence was TTTTHTTHHH, you would say you saw three tails in a row twice.)

$$X_i = \begin{cases} 1 & \text{if 3 tails starts at place } i \\ 0 & \text{else} \end{cases}$$

X = # of times you see 3 tails in a row

$$E(X) = \sum_{i=1}^{8} E[X_i] = 8\,E[X_1] = 8 \times \frac{1}{2^3} = 1$$

6. (5 points) Suppose we have a random variable X with $E[X] = 30$ and $\mathrm{Var}(X) = 10$. Find an upper bound for $P(X > 35)$ using the Chebychev inequality.

$$P(X > 35) = P(X - E(X) > 35 - 30) \leq \frac{10}{5^2} = \frac{10}{25} = \frac{2}{5}$$

7. (5 points) Let X be a random variable with moment generating function $M_X(t) = \frac{1}{1-2t}$ for $t < 1/2$. Let $Y = X^2$. Find $\mathrm{Var}(Y)$.

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = E[X^4] - (E[X^2])^2$$

$$E[X^2] = M^{(2)}(0) \qquad E[X^4] = M^{(4)}(0)$$

$$M^{(1)}(t) = \frac{-1}{(1-2t)^2}(-2) = \frac{2}{(1-2t)^2}$$

$$M''(t) = \frac{-4}{(1-2t)^3}(-2) = \frac{8}{(1-2t)^3} \qquad M''(0) = 8$$

$$M'''(t) = \frac{-24}{(1-2t)^4}(-2) = \frac{48}{(1-2t)^4}$$

$$M^{(iv)}(t) = \frac{-192}{(1-2t)^5}(-2) = \frac{384}{(1-2t)^5}$$

$$M^{(iv)}(0) = 384$$

$$\begin{array}{r} {}^{3}48 \\ \times\ 4 \\ \hline 192 \end{array}$$

$$\mathrm{Var}(Y) = 384 - 8 = 374$$

Table of values for $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998