Math 201: Introduction to Probability

Final Exam

June 22nd, 2023

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- The exam will be 180 minutes long. You will get extra time in the end to upload the exam to Gradescope.
- There are 14 pages.
- A sheet with values of $\Phi(x)$ is provided.
- You may use any formulas from class without proof as long as you state it accurately.
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You do not need to simplify complicated expressions such as $\binom{200}{15}$ or 500!.

QUESTION	VALUE	SCORE
1	0	
2	15(A)	
3	15(A)	
4	10(A)	
5	10(A)	
6	30(B)	
7	30(B)	
8	15(B)	
9	20(B)	
10	20(B)	
11	15(B)	
TOTAL	180	

Value Table of $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

(A) **2. (15(A) points)**

Let X be a continuous random variable with cumulative distribution function (c.d.f.)

$$F_X(x) = \begin{cases} c(2 - e^{-x} - e^{-2x}) & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}$$

where c is a constant.

(a) Find c.

(b) Find the probability density function (p.d.f.), f_X of X.

(c) Find $\mathbb{E}[X]$.

(A) **3. (15(A) points)** Scientists are trying to do an experiment. They know the outcome of the experiment X is random variable with distribution $\mathrm{Ber}(1/3)$. However, due to inaccuracies in their measuring equipment, there is a Gaussian noise parameter $G \sim \mathcal{N}(0,1)$, and what actually gets measured by the equipment is $M = X + G$. Assuming that X and G are independent, what is the probability that $X = 1$ if the measurement M is at most 4?

You may leave your answer in terms of $\Phi(x)$, without evaluating Φ in **this question only**.

(A) **4. (10(A) points)** In the city of Gotham, anyone who likes drinking tea doesn't like drinking anything else. If everyone likes at least one of the beverages tea, coffee, or soda, 20% of people like tea, 75% people like soda, and 60% of people like coffee, then how many people like both coffee and soda?

(A) **5. (10(A) points)**

A national vote was held about pineapple on a pizza. The result says 40% of the population love pineapple on a pizza, while the others hate it.

(a) Triomino's sells pizzas with pineapple on them at \$15 per pizza, and pizzas without pineapple on them at \$14. Let X_n be the money Alfredo's makes from selling n pizzas. Express X_n in terms of a distribution you know.

(b) Find

$$\lim_{n\to\infty} \mathbb{P}(X_n > 14.5n).$$

(B) **6. (30(B) points)**

Suppose (X, Y) are uniformly distributed on the triangular region

$$D = \{(x, y) : x + y \leq 1, x, y \geq 0\}$$

(a) Find the marginal density functions f_X and f_Y.

(b) Find $M_X(t)$.

(c) Using your computations in (b) and (c) or otherwise, find the correlation coefficient $\mathrm{Corr}[X, Y]$.

(d) Are X and Y independent? Remember to justify your answer.

(B) **7. (30(B) points)**

Suppose X and Y are independent random variables with the following moment generating functions (m.g.f.),

$$M_X(t) = \frac{1}{4}e^{-t} + \frac{3}{4}e^{t},$$
$$M_Y(t) = \frac{2}{3} + \frac{1}{3}e^{2t}.$$

(a) Find the joint p.m.f., $p_{X,Y}(x, y)$ and express it as a table.

(b) Let $Z = X + Y$. What is $M_Z(t)$?

(c) Compute the p.m.f. p_Z.

(d) Compute $\mathbb{E}[Z^3] - E[Z^2]$.

(B) **8. (15(B) points)**

Recall that NegBin(k, p) for $k \in \mathbb{N}$ and $0 < p < 1$ is the negative binomial distribution and it is defined as the number of independent Ber(p) trials before one sees k successes.

The p.m.f. of $X \sim \text{NegBin}(k, p)$ is given by

$$p_X(n) = P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k}.$$

In the following parts, it may be useful to recall that if $Y \sim \text{Geom}(p)$, then

$$E[Y] = \frac{1}{p} \qquad \text{Var}[Y] = \frac{1-p}{p^2}.$$

(a) Find a formula for $\mathbb{E}[X]$ and $\text{Var}[X]$. (**Hint:** do not compute this directly! Instead, use the relationship between NegBin and Geom.)

(b) If $X_1 \sim \text{NegBin}(k_1, p)$ and $X_2 \sim \text{NegBin}(k_2, p)$ are independent, then what is the distribution of $X_1 + X_2$? (**Hint:** there is a simple way to do this without using convolutions!)

(B) **9. (20(B) points)**

On average Batman and Robin have to wait 36 hours between the occurrences of two crimes important enough for Commissioner Gordon to light up the Bat-Signal. The Bat-Signal was just lit up, and suppose B is the number of hours from now until the next time the Bat-Signal is lit up.

(a) If you know nothing else about the distribution of B, what upper bound can you provide for the probability that $B > 90$? Clearly state any inequality you use, and explain why the hypotheses apply.

(b) You can assume B is a continuous random variable. Further, B is memoryless – if no Bat-signal worthy crime has occurred in the t hours since the last one, then probability it will take at least s more hours is the same as the probability that he would have taken s hours in the first place. That is,

$$P(B > s + t | B > t) = P(B > s).$$

Can you identify the distribution of B?

(c) Compute $P(B > 90)$ exactly.

(B) **10. (20(B) points)**

Alexander rolls a standard die repeatedly. Let A_j be the indicator random variable for the event that the jth roll is odd, and B_j be the indicator random variable for the event that the jth roll is 5.

(a) Find $\mathbb{E}[A_j B_k]$. (**Hint:** consider the cases $j = k$ and $j \neq k$ separately)

(b) Let X be the number of odd numbers and Y be the number of 5s that show up in n rolls of the die. Find $\mathbb{E}[XY]$.

(c) Compute $\mathrm{Cov}[X, Y]$.

(B) **11. (15(B) points)** No Hobbit in Middle-Earth is taller than 5 feet. You decide to go around randomly asking Hobbits their height to get a good estimate for their average height. Using the law of large numbers, estimate how many Hobbits you need to ask before there is a 99% chance that the average height you sampled differs from the actual average by at most 1 inch? [**Note:** 12 inches is 1 feet.]

Hint: Use Chebyshev's inequality.