# MATH 201 (SUMMER 2023, SESH A2)

## LECTURE 1: 05/15/23

ANURAG SAHAY

OFF HRS: TBA (VIA ZOOM)
[BY APPT.]

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Zoom ID: 979-4693-0650

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

BREAK
~5-15 MIN

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

# COURSE INFORMATION

1. TEXTBOOK & REQUIREMENTS (WORKING WEBCAM!)

2. PREREQUISITES → MATH 162 (OR EQUIV.)
   → MATH 164 IS NOT A PREREQ. BUT IT IS RECOMMENDED.

3. (TENTATIVE) SCHEDULE & COURSE DESCRIPTION

   * ALL LECTURES WILL BE RECORDED
   * OFFICE HOURS WILL NOT BE RECORDED

# COURSE INFORMATION

4. EXAMS

MIDTERM : THURSDAY, JUNE 1st

FINAL EXAM : THURSDAY, JUNE 22nd

THE FINAL WILL HAVE 2 PARTS.

PART A : SAME SYLLABUS AS MIDTERM.

PART B : EVERYTHING AFTER MIDTERM.

5\. HOMEWORK

* ~2 WEBWORK ASSIGNMENTS / WEEK, LOGIN VIA BLACKBOARD
* ~2 WRITTEN ASSIGNMENTS / WEEK, VIA GRADESCOPE
* GRADER: NATHANAEL GRAND

  email: ngrand@ur.rochester.edu

JUSTIFY YOUR ANSWERS!

6\. GRADING

7\. POLICIES: (a) ACADEMIC HONESTY

(b) DISABILITY SUPPORT

(c) ZOOM

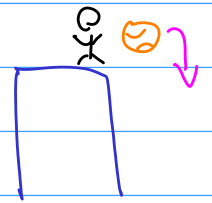
# OTHER NOTES

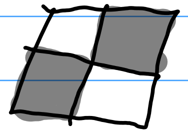
1. (BI) - WEEKLY FEEDBACK FORM
2. WILL UPLOAD LECTURE NOTES & LECTURES
3. MINIMAL USE OF BLACKBOARD
   (ONLY GRADES, WEBWORK LOGIN, & EMAILS)
4. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!
5. DON'T FALL BACK (ONLY $\boxed{6}$ WEEKS!)
   WW DEADLINES ARE FLUID.
   WRITTEN HW ARE RIGID.
6. WE WILL TAKE $\boxed{\text{1-VARIABLE CALC}}$ FOR GRANTED.
   MULTIVARIABLE CALC WILL BE REVIEWED WHEN NECESSARY.

# MOTIVATION

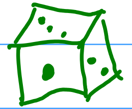
MANY THINGS IN LIFE ARE DETERMINISTIC.

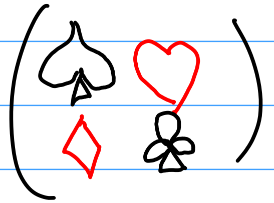
eg.
1. YOUR AGE → YEAR TODAY − YEAR OF BIRTH.
2. THE MOTION OF A BALL THROWN FROM A BUILDING



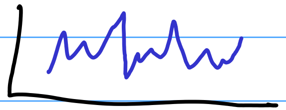
3. A GAME OF CHESS 
4. TAXES

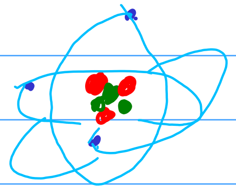
ON THE OTHER HAND MANY THINGS ARE RANDOM!

SIMPLE EXAMPLES: 1 THROWN DICE 

2 PLAYING CARDS 

MORE COMPLICATED EXAMPLES: 3 PRICE OF AN ASSET



4 DYNAMICS OF AN ELECTRON 

GOAL: TO SYSTEMATICALLY STUDY RANDOMNESS USING MATHEMATICS.

PREP: SETS. (APPENDIX B)

COUNTING (APPENDIX C)

# $\boxed{\text{APPENDIX B: SETS}}$

SET $\longrightarrow$ A WELL DEFINED COLLECTION OF $\boxed{\text{OBJECTS}}$ $\to$ ELEMENTS OR POINTS $\omega$

$\Omega$

e.g. 1. SET OF ALL STUDENTS IN THIS COURSE. AFTER DROP DATE.

e.g. 2 $\Omega = \{1, 2, 3, 4, 5, 6\}$ $\longrightarrow$ STANDARD DIE

e.g. 3 $\Omega = \mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{Q}$

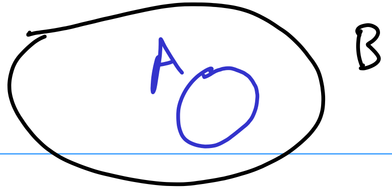
$\omega \in \Omega \longrightarrow \omega$ IS A MEMBER OF $\Omega$

e.g. $1 \in \{1,2,3,4,5,6\}$

$\omega \notin \Omega \longrightarrow \omega$ IS <u>NOT</u> A MEMBER OF $\Omega$

e.g. $0,7 \notin \{1,2,3,4,5,6\}$

## SUBSETS




$A \subseteq B \rightarrow$ (A IS A SUBSET OF B)

(NOT $\in$)

EVERY ELEMENT OF A IS AN ELEMENT OF B.

$A \not\subseteq B \rightarrow$ A IS NOT A SUBSET OF B

e.g.

$\{2, 4, 6\} \subseteq \{1, 2, 3, 4, 5, 6\}$

$\{1, \boxed{7}\} \not\subseteq \{1, 2, 3, 4, 5, 6\}$

# EQUALITY

$\underbrace{A \subseteq B}_{\text{EVERY ELEMENT OF A IS IN B.}}$ & $\underbrace{B \subseteq A}_{\text{EVERY ELEMENT OF B IS IN A.}}$ $\iff$ $\underbrace{A = B}$

$\downarrow$

A & B HAVE EXACTLY THE SAME ELEMENTS

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EMPTY SET $\emptyset$ (SET WITH NO ELEMENTS)

$\rightarrow x \notin \emptyset$ FOR ALL $x$.

$\{ \rightarrow \emptyset \subseteq \Omega$ FOR ALL $\Omega$ $\quad \forall x \in \emptyset \longrightarrow x \in \Omega$

# SET-BUILDER NOTATION

$$\{\omega \in \Omega : \boxed{P(\omega)}\}$$

SOME PROPERTY.

e.g. $\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{\omega \in \Omega : \omega \text{ IS EVEN}\}$

$B = \{\omega \in \Omega : 2 \leq \omega \leq 4\}$

$A = \{2, 4, 6\}$

$B = \{2, 3, 4\}$

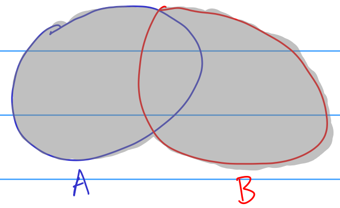
$[A, B \subseteq \Omega]$

$( \text{FIX } \Omega, \quad A, B \subseteq \Omega )$

## UNION

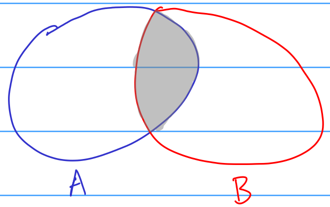
$$A \cup B = \{ \omega \in \Omega : \omega \in A \text{ OR } \omega \in B \}$$

INCLUSIVE



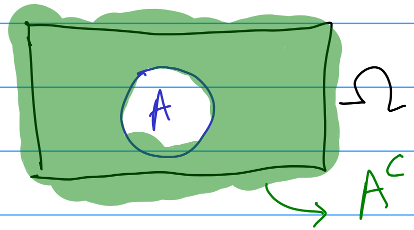

## INTERSECTION

$$A \cap B = \{ \omega \in \Omega : \omega \in A \text{ AND } \omega \in B \}$$




## COMPLEMENT

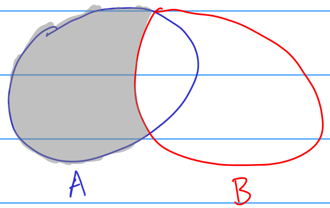
$$A^c = \{ \omega \in \Omega : \omega \notin A \}$$




## (SET) DIFFERENCE

$$A \setminus B = \{ \omega \in \Omega : \omega \in A \text{ AND } \omega \notin B \}$$

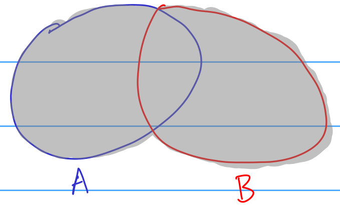
$A - B$



$\Omega = \{1,2,3,4,5,6\}$ $\quad A = \{2,4,6\}$ $\quad B = \{2,3,4\}$

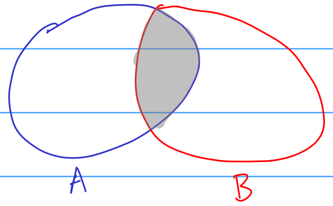
## UNION

$A \cup B = \{2,3,4,6\}$



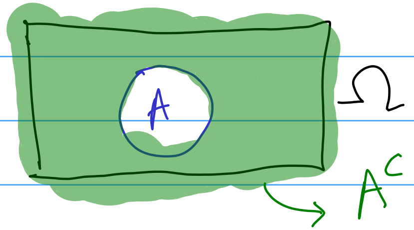

## INTERSECTION

$A \cap B = \{2,4\}$



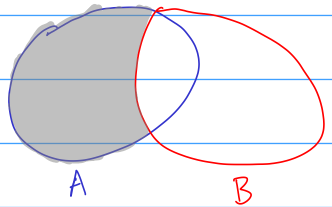

## COMPLEMENT

$A^c = \{1,3,5\}$




## (SET) DIFFERENCE

$A \setminus B = \{6\}$, $B \setminus A = \{3\}$.



# ITERATED UNION / INTERSECTION

RECALL:

$$\sum_{j=1}^{n} a_j = a_1 + a_2 + \cdots + a_n$$

lly:

$$\bigcup_{j=1}^{n} A_j = A_1 \cup A_2 \cup \cdots \cup A_n = \{\omega \in \Omega : \omega \in A_j \text{ FOR SOME } j\}$$

$$\bigcap_{j=1}^{n} A_j = A_1 \cap A_2 \cap \cdots \cap A_n = \{\omega \in \Omega : \omega \in A_j \text{ FOR ALL } j\}$$

e.g. $\Omega = \{1, 2, 3, 4, 5, 6\}$

$A_1 = \{2, 4, 6\}$ $\quad$ $A_2 = \{2, \boxed{3}, 4\}$

$A_3 = \{2, 4, 5\}$ $\quad$ $A_4 = \{2, 6\}$

$$\bigcup_{j=1}^{4} A_j = \{2, 3, 4, 5, 6\}$$

$$\bigcap_{j=1}^{4} A_j = \{2\}$$

$$\bigcup_{j} A_j$$

$$\bigcap_{j=1}^{\infty} A_j = [A_1 \cap A_2 \cap A_3 \quad \cdots \quad ]$$

$\Omega \supseteq A_j$

# DE MORGAN'S LAW

$$\left(\bigcap_j A_j\right)^c = \bigcup_j A_j^c \qquad \left[(A \cap B)^c = A^c \cup B^c\right]$$

$$\left(\bigcup_j A_j\right)^c = \bigcap_j A_j^c \qquad \left[(A \cup B)^c = A^c \cap B^c\right]$$

Pf. $(A \cup B)^c = A^c \cap B^c$ [OTHERS ARE SIMILAR]

① $(A \cup B)^c \subseteq A^c \cap B^c$

$x \in (A \cup B)^c \iff x \notin A \cup B$

$\iff x \notin A$ AND $x \notin B$

$\iff x \in A^c$ AND $x \in B^c$

$\iff x \in A^c \cap B^c$

② $A^c \cap B^c \subseteq (A \cup B)^c$

BACK AT

10:30 AM ET

# APPENDIX C: COUNTING

FOR A SET $\Omega$,

$\#\Omega \longrightarrow \underbrace{\text{CARDINALITY}}_{\text{\# OF ELEMENTS.}}$ OF $\Omega$

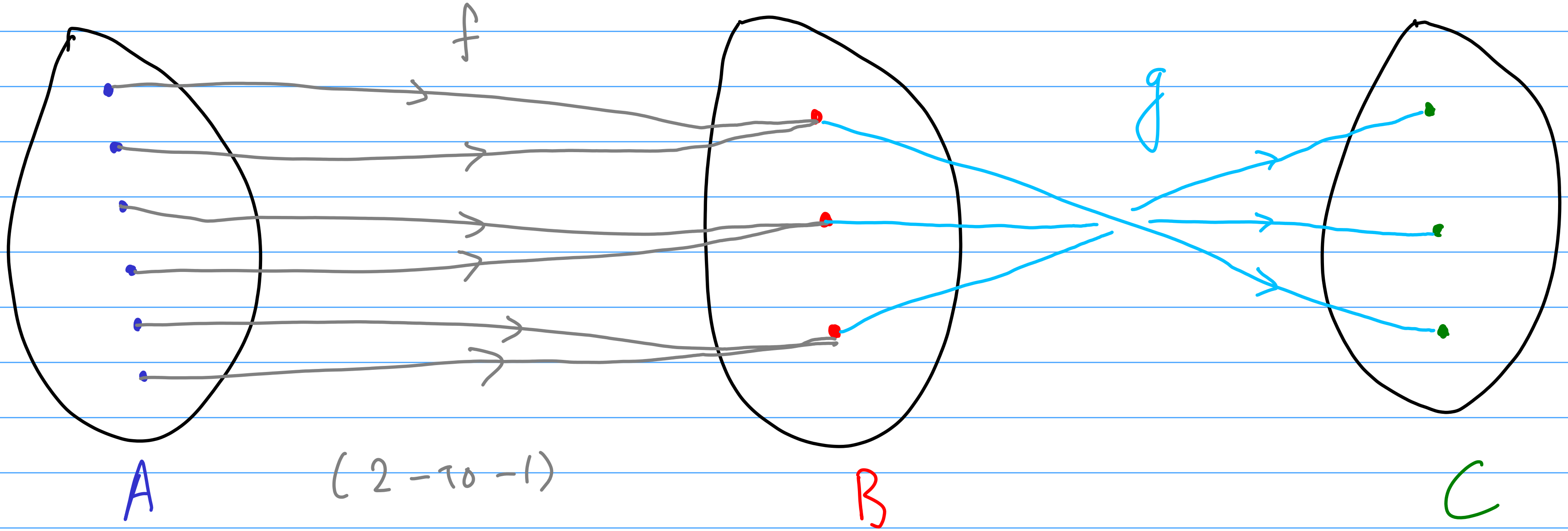
e.g. IF $\Omega = \{1, 2, 3, 4, 5, 6\}$

$$\#\Omega = 6$$

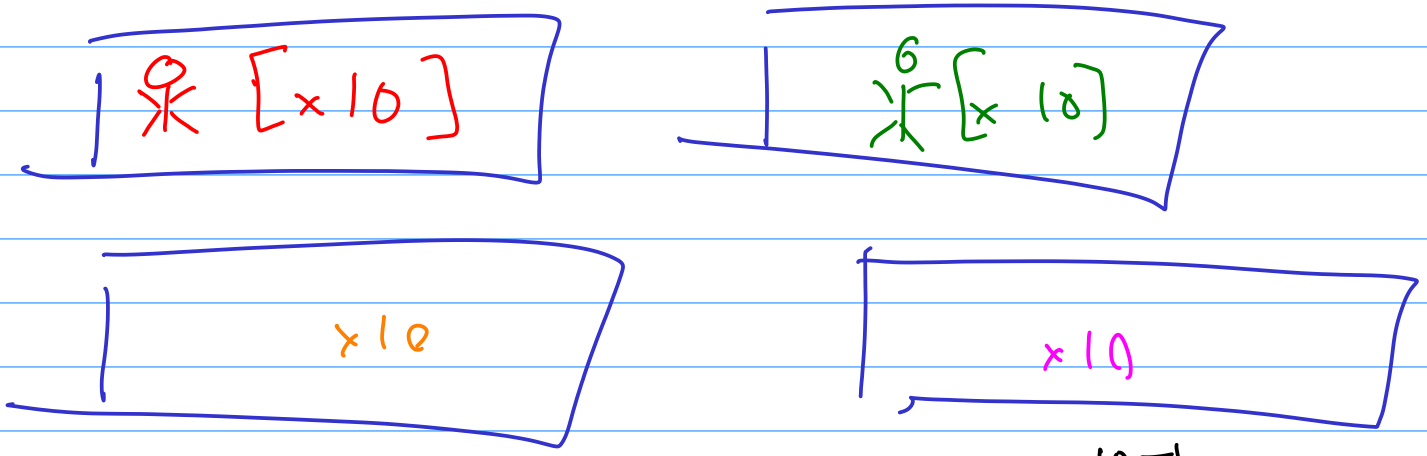
# PRINCIPLES OF COUNTING.

(USUALLY $k=1$)

**Fact C.1.** Let $A$ and $B$ be finite sets and $k$ a positive integer. Assume that there is a function $f$ from $A$ onto $B$ so that each element of $B$ is the image of exactly $k$ elements of $A$. (Such a function is called *k-to-one*.) Then $\#A = k \cdot \#B$.



**Example C.2.** Four fully loaded 10-seater vans transported people to the picnic. How many people were transported? Clearly the answer is $10 \cdot 4 = 40$. Here $A$ is the set of people, $B$ is the set of vans, and $f$ maps a person to the van she rides in. $\#A = 40$, $\#B = 4$, and $f$ is a 10-to-one function from $A$ onto $B$. ▲




B = VANS

A = PEOPLE

$f : A \xrightarrow{10-1} B$

$x \longmapsto$ VAN $x$ IS IN.

$\#B = 4$

$\#A = 10 \times 4 = 40$

$(a_1, \dots, a_n) \to$ ORDER MATTERS

$26 = 21 + 5$

**Fact C.3.** Suppose that a set of $n$-tuples $(a_1, \dots, a_n)$ obeys these rules.

(i) There are $r_1$ choices for the first entry $a_1$.

(ii) Once the first $k$ entries $a_1, \dots, a_k$ have been chosen, the number of alternatives for the next entry $a_{k+1}$ is $r_{k+1}$, regardless of the previous choices.

Then the total number of $n$-tuples is the product $r_1 \cdot r_2 \cdot r_3 \cdots r_n$.

CONSONANTS.

21 __ __

VOWEL

$(a_1, a_2, a_3)$

(GENERAL MULTIPLICATION PRINCIPLE)

Pf:

$\underline{a_1}$ $\underline{a_2}$ $\underline{a_3}$ .... __ __ $\underline{a_n}$

$r_1$ $r_2$ $r_3$

$r_1 r_2 \cdots r_n$

Pf : INDUCTION.

BASE : $n = 1 \rightarrow$ TOTAL # $= r_1$

SUPPOSE THM IS TRUE FOR $n = 1, \dots, m$

(e.g. $m = 2$, $n = 1$)

INDUCTIVE $A = \{(a_1, \dots, a_{m+1}) : \text{VALID CHOICE}\}$

$B = \{(a_1, \dots, a_m) : \text{VALID CHOICE}\}$

$\# B = r_1 \cdot r_2 \cdots r_m$

$$f : A \longrightarrow B$$

$$(a_1, \dots a_m, a_{m+1}) \longmapsto (a_1, \dots, a_m)$$

$f$ IS $n_{m+1}$ -TO -ONE

$$\# A = (\# B) \cdot (n_{m+1})$$

$$= (n_1 \cdots n_m) \cdot n_{m+1}$$

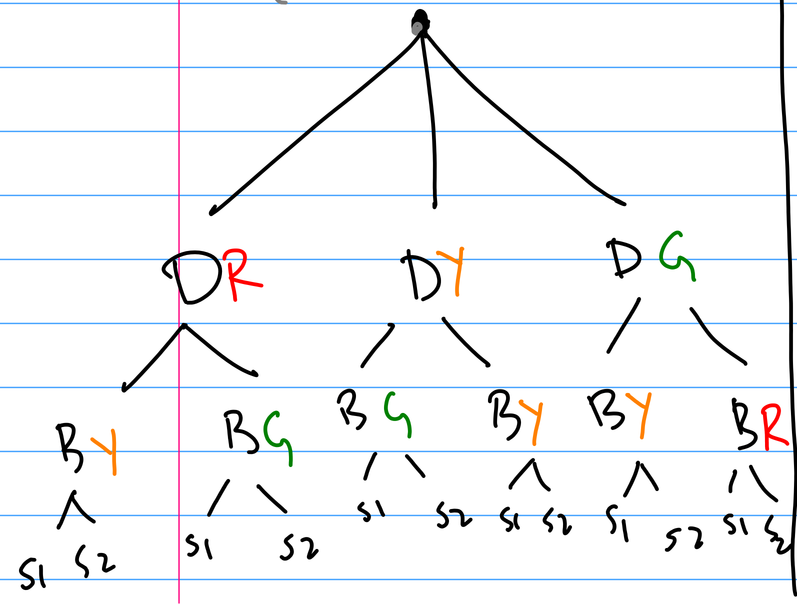
**Example C.4.** To dress up for school in the morning, Joyce chooses from 3 dresses (red, yellow, or green), 3 blouses (also red, yellow, or green), and 2 pairs of shoes. She refuses to wear a dress and a blouse of matching colors. How many different outfits can she choose from?

D C

B C

S1, S2.

(DECISION TREE)




$(c_1, c_2, c_3)$

DRESS BLOUSE SHOES

(R, Y, ___)

3 2 2.

$3 \times 2 \times 2 = 12$

# §1.1 SAMPLE SPACES & PROBABILITIES

MATHEMATICAL MODEL FOR RANDOMNESS

↳ KOLMOGOROV'S AXIOMS

3 INGREDIENTS : $\underbrace{(\Omega, \mathcal{F}, P)}_{\text{PROBABILITY SPACE}}$

$\Omega$ :

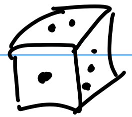
- The **sample space** $\Omega$ is the set of all the possible outcomes of the experiment. Elements of $\Omega$ are called **sample points** and typically denoted by $\omega$.

OUTCOMES.

e.g.

1. COIN-FLIP

$$\Omega = \{\text{HEADS}, \text{TAILS}\}$$

2. DIE. ROLLING 

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$\mathcal{F}$ :

- Subsets of $\Omega$ are called **events**. The collection of events in $\Omega$ is denoted by $\mathcal{F}$. ♣

$E \subseteq \Omega \longrightarrow$ E IS AN EVENT.

e.g.

OUTCOME OF A DIE ROLL IS EVEN

$\Omega = \{1, 2, 3, 4, 5, 6\}$ $\qquad A = \{2, 4, 6\} = \{\omega \in \Omega : \omega \text{ IS EVEN}\}$

OUTCOME OF A DIE ROLL IS $< 4$.

$A = \{1, 2, 3\} \rightarrow$ OUTCOME IS $\leq 3$

$A \in \mathcal{F}$

$P(A) \in \mathbb{R}$

- The **probability measure** (also called **probability distribution** or simply **probability**) $P$ is a function from $\mathcal{F}$ into the real numbers. Each event $A$ has a probability $P(A)$, and $P$ satisfies the following axioms.
  (i) $0 \leq P(A) \leq 1$ for each event $A$.
  (ii) $P(\Omega) = 1$ and $P(\varnothing) = 0$.
  (iii) If $A_1, A_2, A_3, \ldots$ is a sequence of pairwise disjoint events then

$A_i \cap A_j = \phi \quad (i \neq j)$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \tag{1.1}$$

SUBTLE.

$A_1 = \{H\}$

$A_2 = \{T\}$

$A_j = \phi$ o.w.

e.g. :

$\Omega = \{H, T\}$

$P(\{H, T\}) = 1$

$P(\{H\}) = P(\{T\}) = 1/2$

$P(\phi) = 0$

"FAIR COIN"

$P(\{H\} \cup \{T\}) \overset{?}{=} \underbrace{P(\{H\})}_{= 1/2} + \underbrace{P(\{T\})}_{= 1/2}$

$1 = $

N.B. $P(\{x\}) \equiv P(x)$

**Fact 1.2.** If $A_1, A_2, \ldots, A_n$ are pairwise disjoint events then

$$P(A_1 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n). \tag{1.2}$$

__Pf__: SET $A_{n+1} = A_{n+2} = A_{n+3} = \cdots = \emptyset$

$P(A_{n+1}) = P(A_{n+2}) = \cdots = 0$

"FAIR DIE"

__e.g.__ DIE-ROLL $\Omega = \{1,2,3,4,5,6\}$

$P(1) = P(2) = P(3) = P(4) = P(5) = P(6) = \frac{1}{6}$

$P(\text{EVEN}) = P(\{2,4,6\}) = P(\{2\} \cup \{4\} \cup \{6\})$

$= P(2) + P(4) + P(6) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

UNFAIR / LOADED DIE

$\Omega = \{1, \ldots, 6\}$

e.g. $\widetilde{P}\{1\} = \widetilde{P}\{2\} = \widetilde{P}\{3\} = \widetilde{P}\{4\} = \widetilde{P}\{5\} = \frac{1}{7}$ and $\widetilde{P}\{6\} = \frac{2}{7}$. (LOADED TO 6)

$\widetilde{P}(\text{EVEN}) = \widetilde{P}(2) + \widetilde{P}(4) + \widetilde{P}(6) = 1/7 + 1/7 + 2/7 = 4/7 > 1/2$

$Q\{1\} = \frac{1}{6},\ Q\{2\} = \frac{2}{6},\ Q\{3\} = \frac{1}{6},\ Q\{4\} = \frac{1}{6},\ Q\{5\} = 0,\ Q\{6\} = \frac{1}{6}.$ (5 REPLACED BY 2)

$Q(\text{EVEN}) = 4/6 = 2/3$

N.B.: ① $P, \widetilde{P}, Q$ ARE ON THE SAME $\Omega$

② $Q(5) = 0$ !