MATH 201 (SUMMER 2023, SESH A2)

LECTURE 10 : 05/31/23

ANURAG SAHAY

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE: https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. MIDTERM 1 : THURS, 1st JUNE (IN-CLASS / REVIEW TODAY)
2. OFFICE HOURS : AFTER CLASS TODAY
3. UPCOMING DEADLINES :
   a. WW 05 – TODAY, MAY 31st
   b. HW 05 – SAT, JUNE 3rd
   c. WW 06 – TUES, JUNE 6th

   $\subseteq$ { SYLLABUS FOR EXAM }
4. HW 1 IS GRADED.
   → SOLNS TO HW 1 THROUGH 3.
   → SOLNS TO HW YOU HAVEN'T SUBMITTED WILL NOT BE PROVIDED
5. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 4.5 EXPONENTIAL DISTRIBUTION

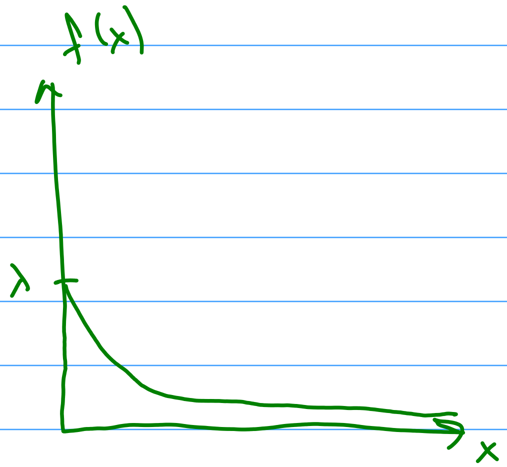
→ PRINCIPLE: CONTINUOUS ANALOGUE OF Geom(p)

$\lambda \approx p$

Definition 4.26. Let $0 < \lambda < \infty$. A random variable X has the **exponential distribution** with $\underbrace{\text{parameter } \lambda}_{\text{RATE}}$ if X has density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{4.16}$$

on the real line. Abbreviate this by $X \sim \operatorname{Exp}(\lambda)$. The $\operatorname{Exp}(\lambda)$ distribution is also called the *exponential distribution with rate* λ.



$\lambda > 0$ $\quad e^{-\lambda x} \to 0$ As $x \to \infty$

① $\int p.d.f \quad (=1?)$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \underbrace{\lambda e^{\overbrace{-\lambda x}^{y}}\,dx}_{-dy} = -e^{-\lambda x}\Big]_{x=0}^{x=\infty} = -\left(-e^{-\lambda \cdot 0}\right) = 1$$

② MEAN $\left(= 1/\lambda\right)$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \underbrace{x}_{u} \underbrace{\lambda e^{-\lambda x}\,dx}_{dv} = \cancel{-xe^{-\lambda x}\Big]_0^{\infty}} - \int_0^{\infty} \left(-e^{-\lambda x}\right) dx$$

$$= \int_0^{\infty} e^{-\lambda x}\,dx = \frac{-e^{-\lambda x}}{\lambda}\Big]_0^{\infty} = \frac{1}{\lambda}$$

CHECK $\lim_{x \to \infty} xe^{-\lambda x} = 0$

③ VARIANCE. $\left(= 1/\lambda^2\right)$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$\mathbb{E}(X)$

$$= \cancel{-x^2 e^{-\lambda x}\Big]_0^{\infty}} - \int_0^{\infty} \left(-2x e^{-\lambda x}\right)dx$$

$$\lim_{x \to \infty} x^2 e^{-\lambda x} = 0$$

$$= \int_0^{\infty} 2x e^{-\lambda x}\,dx = \frac{2}{\lambda} \boxed{\int_0^{\infty} x \cdot \underbrace{\lambda e^{-\lambda x}}_{\text{p.d.f.}}\,dx}$$

$$\mathbb{E}(X^2) = \frac{2}{\lambda} \mathbb{E}(X) = \frac{2}{\lambda} \cdot \frac{1}{\lambda} = \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

(4) DETERMINED BY TAIL:

$$\underbrace{P(X > t)}_{\text{RIGHT TAIL OF DISTRIBUTION}} = 1 \quad \text{IF } t < 0$$

$$\left(\because\ P(X \leq 0) = \int_{-\infty}^{0} f(x)\,dx = 0\right)$$

IF $t > 0$,

$$P(X > t) = \int_{t}^{\infty} f(x)\,dx = \int_{t}^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big]_{t}^{\infty}$$

$$= e^{-\lambda t}$$

$$P(X > t) = \begin{cases} 1 & t \leq 0 \\ e^{-\lambda t} & t > 0 \end{cases}$$

(5) c.d.f.

$$F(t) = P(X \le t) = \int_{-\infty}^{t} f(x)\,dx = \begin{cases} 0 & \text{IF } t < 0 \\ 1 - e^{-\lambda t} & \text{IF } t \geq 0 \end{cases}$$

$$\{X \le t\} = \{X > t\}^c$$

$$F(t) = P(X \le t) = 1 - P(X > t) = \begin{cases} 0 & \text{IF } t < 0 \\ 1 - e^{-\lambda t} & \text{IF } t \geq 0 \end{cases}$$

$\lambda = 1/2$

Example 4.27. Let $X \sim \text{Exp}(1/2)$. Find $P(X > \frac{7}{2})$. Also, find the median of X, that is, the value m for which $\boxed{P(X \le m) = \frac{1}{2}}$.

$$\underbrace{P(X > 7/2)}_{\text{TAIL}} = \int_{7/2}^{\infty} \frac{1}{2} \cdot e^{-\frac{1}{2}x} \, dx$$

$$P(x > 7/2) = e^{-\frac{1}{2} \cdot \frac{7}{2}} = e^{-7/4}$$

$$P(X \le m) = F_X(m) = \begin{cases} 0 & \cancel{\text{IF } m < 0} \\ 1 - e^{-\frac{1}{2}m} & \text{IF } m \geq 0 \end{cases}$$

$$F_X(m) = 1/2 = 1 - e^{-\frac{1}{2}m} \Rightarrow e^{-m/2} = 1/2 \Rightarrow m = \boxed{2 \ln 2}$$

MEDIAN.

Example 4.28. Let $X \sim \mathrm{Exp}(\lambda)$ and $Z = X/2$. Find $P(Z > t)$ for $t > 0$.

$$P(Z>t) = P\left(X/2 > t\right) = \underbrace{P(X > 2t)}_{\text{TAIL OF } X}$$

$$= e^{-\lambda(2t)} = e^{-(2\lambda)t}$$

$\lambda' = 2\lambda$

$$= \underbrace{e^{-\lambda' t}}_{\text{TAIL OF } \mathrm{Exp}(\lambda')}$$

Q: DIST. OF Z?

$\rightarrow \quad Z \sim \mathrm{Exp}(2\lambda)$

GEN. FACT.

$$X \sim \mathrm{Exp}(\lambda)$$

$$Z = \alpha X$$

$$Z \sim \mathrm{Exp}\left(\frac{\lambda}{\alpha}\right)$$

$X \sim \mathrm{Exp}(\lambda) \longrightarrow$ WAITING TIME

$X = t \longrightarrow$ FIRST PERSON COMES IN AT TIME t.

Fact 4.31. (Memoryless property of the exponential distribution) Suppose that $X \sim \mathrm{Exp}(\lambda)$. Then for any $s, t > 0$ we have

$$P(X > t + s \mid X > t) = P(X > s). \tag{4.19}$$

↳ TIME YOU STILL HAVE TO WAIT DOESN'T DEPEND ON THE TIME YOU HAVE ALREADY WAITED.

NOTE: SIMILAR TO Geom(p) [SEEN IN H.W. 3]

IN FACT: $\mathrm{Exp}(\lambda)$ IS THE ONLY MEMORYLESS & CONTINUOUS R.V.

PF IN BOOK

Pf of MEMORYLESS

$$P(X > s+t \mid X > t) = \frac{P(\overbrace{\{X>s+t\} \cap \{X>t\}}^{X>s+t})}{P(X>t)}$$

$X > s+t \Rightarrow X > t$ (i.e. $\{X>s+t\} \subseteq \{X>t\}$)

$$P(X>s+t \mid X>t) = \frac{P(X>s+t) \rightarrow \text{TAIL}}{P(X>t) \rightarrow \text{TAIL}} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda(t)}} = \frac{e^{-\lambda s} \cdot \cancel{e^{-\lambda t}}}{\cancel{e^{-\lambda t}}}$$

$$= e^{-\lambda s}$$

$$P(X > s+t \mid X > t) = \underbrace{e^{-\lambda s}}_{\text{TAIL}} = P(X > s)$$

$\boxtimes$

Example 4.32. Animals of the forest need to cross a remote highway. From experience they know that, from the moment someone arrives at the roadside, the time till the next car is an exponential random variable with expected value 30 minutes. The turtle needs 10 minutes to cross the road. (a) What is the probability that the turtle can cross the road safely? (b) Now suppose that when the turtle arrives at the roadside, the fox tells her that he has been there already 5 minutes without seeing a car go by. What is the probability now that the turtle can cross safely?

X = TIME TILL NEXT CAR ARRIVES

$\frac{1}{\lambda} = \mathbb{E}(X) = 30$, $X \sim Exp(\lambda)$ $\Rightarrow$ $\lambda = 1/30$

(a) $P(X > 10) = e^{-\lambda t} = e^{-\frac{1}{30} \cdot 10} = e^{-1/3} \simeq 0.7165$

Y = TIME WHEN CAR GOES BY AFTER FOX ARRIVE.

$Y > 10 + 5$ $\qquad\qquad$ $Y > 5$

$Y \sim \mathrm{Exp}(1/30)$

SAFELY ARRIVING $= P(Y > 15 \mid Y > 5)$

$= P(Y > 10) = e^{-1/3}$

RETURN AT

10:00 AM

SAMPLE MIDTERM

$f : \{0,1\}^2 \to \{0,1\}$

Q5. (b)

$$f(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

$X, Y \sim Ber(1/2)$, INDEPENDENT, $Z = f(X, Y)$

ARE THEY MUTUALLY INDEPENDENT?

$P(X=0) = P(Y=0) = P(Z=0) = 1/2$, $P(X=1) = P(Y=1) = P(Z=1) = 1/2$

X, Y, Z

1. PAIR-WISE IND.

2. $P(X = k_1, Y = k_2, Z = k_3) \overset{?}{=} P(X = k_1) \cdot P(Y = k_2) \cdot P(Z = k_3)$

$X = 0, Y = 0 \Rightarrow Z = 0 \quad (\because f(0,0) = 0)$

$L.H.S. = P(X = 0, Y = 0, Z = 1) = 0$ $\neq$

$R.H.S. = P(X=0) \cdot P(Y=0) \cdot P(Z=1) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$

$$k_1 = k_2 = k_3 = 0$$

$$L.H.S. = P(X=0, Y=0, Z=0) = P(X=0, Y=0)$$

$$= P(X=0) \cdot P(Y=0) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$$R.H.S. = P(X=0) \cdot P(Y=0) \cdot P(Z=0) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$\neq$

DERIVATION OF VARIANCE FOR Bin(n, p)

Q3. (c)

$$\mathrm{Var}(X) = n \cdot p\,(1-p)$$

$(q := 1-p)$

$$E(X^2) = \sum_{k} k^2 \; P(X = k)$$

$$= \sum_{k=0}^{n} k^2 \binom{n}{k} \cdot p^k \cdot q^{n-k}$$

$j = k-1$

$$= \sum_{k=1}^{n} k^{\cancel{2}} \cdot \frac{n!}{(k-1)!\,\cancel{k!} \cdot (n-k)!} \cdot p^k \cdot q^{n-k}$$

$$np \quad \sum_{j=0}^{n-1} (j+1) \cdot \frac{n!}{\underbrace{j! \cdot (n-1-j)!}_{\approx \binom{n-1}{j}}} p^{j+1} q^{n-1-j}$$

$$E(\mathrm{Bin}(n-1, p)) = (n-1)p$$

$$= np \sum_{j=0}^{n-1} (j+1) \binom{n-1}{j} p^j q^{n-1-j}$$

$$= np \left[\sum_{j=0}^{n-1} j \cdot \binom{n-1}{j} p^j q^{n-1-j} \right] + np \left[\underbrace{\sum_{j=0}^{n-1} \binom{n-1}{j} p^j q^{n-1-j}}_{(p+q)^{n-1} = 1} \right]$$

$$= np\left[(n-1)\cdot p\right] + np$$

$$\mathbb{E}(X^2) = n^2p^2 - np^2 + np$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \left(\cancel{n^2p^2} - np^2 + np\right) - \cancel{(np)^2}$$

$$= np - np^2 = np\left[1-p\right] = npq$$

DERIVATION OF Exp AS A LIMIT OF
Geom. (WAITING TIMES)

CONSIDER A RANDOM VARIABLE $X \in [0, \infty)$ WHICH REPRESENTS TIME OF ARRIVAL OF FIRST CUSTOMER AT A SHOP.

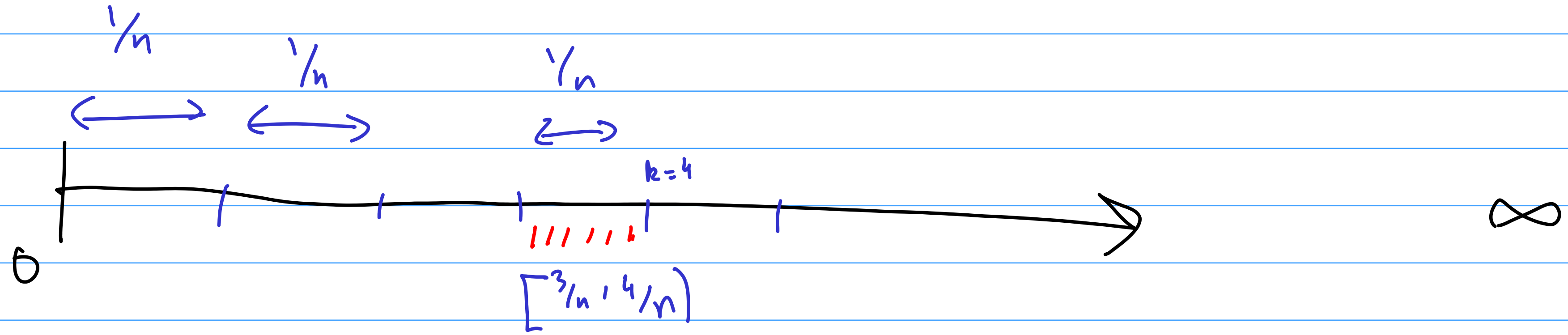
SUPPOSE THAT :

① $P\left(\begin{array}{l}\text{ARRIVAL TIME OF 1st CUST.}\\ \text{IS IN AN INTERVAL OF LENGTH } \delta\end{array}\right) \approx \boxed{\lambda}\,\delta \quad (\delta \to 0)$ FIXED.

② GIVEN n CUSTOMERS AND n DISJOINT INT.

THE EVENTS C_j ARRIVES IN I_j ARE MUTUALLY INDEPENDENT, $(j = 1, \dots, n)$

LET $n \to \infty$, AND DISCRETIZE $[0, \infty)$ INTO INTERVALS OF LENGTH $1/n$



$\{T_n = k/n\}$ = 1st ARRIVAL IN $\left[\frac{k-1}{n}, \frac{k}{n}\right)$

$P(T_n = k/n) = P\Big($ NO ARRIVED IN $\left[\frac{j-1}{n}, \frac{j}{n}\right)$ FOR $j < k$, SOMEONE ARRIVED IN $\left[\frac{k-1}{n}, \frac{k}{n}\right)\Big)$

($T_n \to$ DISCRETIZED X)

$$= \left(\prod_{j=1}^{k-1} P\left(\text{NO ONE ARRIVED IN } \left[\frac{j-1}{n}, \frac{j}{n}\right) \right) \right) \cdot P\left(\text{SOMEONE ARRIVED AT } \left[\frac{k-1}{n}, \frac{k}{n}\right) \right)$$

$$\approx \underbrace{\left(1 - \frac{\lambda}{n}\right)^{k-1} \cdot \left(\frac{\lambda}{n}\right)}_{P(X = k), \quad X \sim \text{Geom}\left(\frac{\lambda}{n}\right)}$$

$$T_n = k/n \iff n T_n = k$$

$$n T_n \approx \text{Geom}\left(\frac{\lambda}{n}\right)$$

$\therefore \quad n T_n \approx \mathrm{Geom}\left(\frac{\lambda}{n}\right)$

WE WILL SHOW THAT AS $n \to \infty$,

$T_n \longrightarrow \mathrm{Exp}(\lambda)$

PROB.

Theorem 4.33. *Fix $\lambda > 0$. Consider n large enough so that $\lambda/n < 1$. Suppose that for each large enough n, the random variable T_n satisfies $nT_n \sim \text{Geom}(\lambda/n)$. Then*

$$\lim_{n\to\infty} P(T_n > t) = \boxed{e^{-\lambda t}} \qquad \textit{for all nonnegative real } t. \tag{4.21}$$

$P(X > t)$ IF $X \sim \text{Exp}(\lambda)$

$$\text{Exp}(\lambda) \approx \frac{1}{n}\,\text{Geom}\left(\frac{\lambda}{n}\right)$$

$$T_n \in \frac{1}{n}\mathbb{N} = \left\{\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots\right\}$$

$$P(T_n > t) = \sum_{k=nt}^{\infty} P(nT_n = k) = \sum_{k=nt}^{\infty} \left(1 - \frac{\lambda}{n}\right)^{k-1} \cdot \left(\frac{\lambda}{n}\right)$$

(GEOM. SERIES)

$$= \frac{\left(1-\frac{\lambda}{n}\right)^{nt-1} \cdot \cancel{\frac{\lambda}{n}}}{\cancel{1-\left(1-\frac{\lambda}{n}\right)}}$$

$$= \left(1-\frac{\lambda}{n}\right)^{nt-1}$$

$$\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt-1} = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{nt} \cdot \left(1-\cancel{\frac{\lambda}{n}}\right)^{-1}$$

$$= \left(\overbrace{\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^{n}}^{e^{-\lambda}}\right)^{t} = \left(e^{-\lambda}\right)^{t} = e^{-\lambda t}$$