MATH 201 (SUMMER 2023, SESH A2)

LECTURE 11: 06/05/23

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OFF HRS: BY APPT (VIA ZOOM)

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID: 979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. MIDTERM IS GRADED!

   a. REGRADES BY WED, JUNE 7th AT 11 PM ET.

   b. ROUGH GRADING : $\}\to$ IF ONLY THE MIDTERM.

$\geq$	100/115	—	A / A-
$\geq$	90/115	—	B+/B/B-
$\geq$	75/115	—	C+/C/C-
$\geq$	50/115	—	D+/D/D-
$<$	50/115	—	E

PLEASE SEND ME AN EMAIL.

NOT INDICATIVE OF ACTUAL GRADES.

   c. STATS : MAX – 112, MEAN – 91.21, MEDIAN – 97 (OUT OF 115)

ANNOUNCEMENTS

2. OFFICE HOURS : TR : 11:15 AM – 12:15 PM

3. UPCOMING DEADLINES :
   a. HW 5 TODAY → POSTPONED FROM LAST WEEK.
   b. HW 6 / WW 6 – WED
   c. WW 7 – SAT
   d. HW 7 – SUN → UPLOADED RECENTLY.

4. PLEASE FILL OUT MID-SEM FEEDBACK. → ANONYMOUS
   → OPTIONAL

5. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 6.1 JOINT DISTRIBUTION OF DISCRETE R.V.s

RECALL: $\{P(X \in B) : B \subseteq \mathbb{R}, B \text{ "REASONABLE"}\}$

X_1, X_2 $\quad X_1 \in B_1$ & $X_2 \in B_2$

$\{P(X_1 \in B_1, X_2 \in B_2) : B_1, B_2 \subseteq \mathbb{R}, \text{ "REASONABLE"}\}$

$\ldots \{P(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n) : B_1, \ldots, B_n \subseteq \mathbb{R}\}$ → JOINT DIST. OF n R.Vs.

$p: \mathbb{R}^n \longrightarrow [0,1]$

$\hookrightarrow (k_1, k_2, \ldots, k_n)$ s.t. $P(X_j = k_j) \neq 0$

Definition 6.1. Let $X_1, X_2, \ldots, X_n$ be discrete random variables, all defined on the same sample space. Their **joint probability mass function** is defined by

$p = p_{X_1, X_2, \ldots, X_n}$

$$p(k_1, k_2, \ldots, k_n) = P(X_1 = k_1, X_2 = k_2, \ldots, X_n = k_n)$$

for all possible values $k_1, k_2, \ldots, k_n$ of $X_1, X_2, \ldots, X_n$.

ANY EVENT INVOLVING $X_1, \ldots, X_n$ SIMULTANEOUSLY CAN BE DESCRIBED USING p.

① $p(k_1, k_2, \ldots, k_n) \geq 0$ (p.m.f. ≥ 0)

n-fold sum.

② $$\sum_{\boxed{k_1,k_2,\ldots,k_n}} p_{X_1,X_2,\ldots,X_n}(k_1,k_2,\ldots,k_n) = 1.$$

EXACTLY ONE OF $(X_1, \ldots X_n) = (k_1, \ldots, k_n)$ MUST HAVE OCCURRED

RUNS OVER EVERY VALUE k_j TAKEN BY X_j

$\sum_k p_X(k) = 1$

$g : \mathbb{R}^n \longrightarrow X$

$g(X_1, \dots, X_n)$ IS A R.V.

3

Fact 6.2. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of an n-vector. If $X_1, \dots, X_n$ are discrete random variables with joint probability mass function p then

$$E[g(X_1, \dots, X_n)] = \sum_{k_1, \dots, k_n} g(k_1, \dots, k_n)\, p(k_1, \dots, k_n) \tag{6.1}$$

provided the sum is well defined.

(COMPARE: $\mathbb{E}(g(X)) = \sum_k g(k)\, p_X(k)$)

Pf: IN THE BOOK

$$\{g(X_1, \dots, X_n) = \ell\} = \bigcup_{k_j :\, g(k_1, \dots, k_n) = \ell} \{X_1 = k_1, X_2 = k_2, \dots, X_n = k_n\}$$

Example 6.3. Flip a fair coin three times. Let X be the number of tails in the first flip and Y the total number of tails observed. ~~The possible values of X are $\{0, 1\}$ and the possible values of Y are $\{0, 1, 2, 3\}$~~. Let us first record which outcomes of the experiment lead to particular (X, Y)-values, with H for heads and T for tails.

$$\Omega = \{H, T\}^3$$

$$= \underline{H/T}\ \underline{H/T}\ \underline{H/T}$$

		Y			
		0	1	2	3
X	0	$\boxed{\text{HHH}}$	$\boxed{\text{HTH, HHT}}$	HTT	✗
	1	?	THH	THT, TTH	TTT

$X \to \{0,1\}$ - VALUED

$Y \to \{0,1,2,3\}$ - VALUED

$X = 1 \Rightarrow Y \neq 0$

i.e. $P(X=1, Y=0) = 0$

Example 6.3. Flip a fair coin three times. Let X be the number of tails in the first flip and Y the total number of tails observed. The possible values of X are $\{0, 1\}$ and the possible values of Y are $\{0, 1, 2, 3\}$. Let us first record which outcomes of the experiment lead to particular (X, Y)-values, with H for heads and T for tails.

$\omega \in \Omega$

$P(\omega) = \frac{1}{2^3} = \frac{1}{8}$

$P(\omega_1, \omega_2) = \frac{2}{8} = \frac{1}{4}$

$X \backslash Y$	0	1	2	3
0	$\frac{1}{8}$ HHH	HTH, $\frac{1}{4}$ HHT	HTT $\frac{1}{8}$	0
1	0	THH $\frac{1}{8}$	THT, $\frac{1}{4}$ TTH	TTT $\frac{1}{8}$

JOINT p.m.f. $\quad p_{X,Y}(k, l)$

$k \in \{0, 1\}$

$l \in \{0, 1, 2, 3\}$

$p_{X,Y}(1,0) = p_{X,Y}(0,3) = 0$

$p_{X,Y}(1,1) = \frac{1}{8}$

$X \rightsquigarrow$ FIRST FLIP

$Y \to$ TOTAL

Suppose each tails earns you 3 dollars, and if the first flip is tails, each reward is doubled. This reward is encoded by the function $g(x, y) = 3(1 + x)y$. The expected reward is ?

HTT $\longrightarrow$ \$6

THT $\longrightarrow$ \$12

HHH $\longrightarrow$ \$0

$$g(X,Y) = (3Y)\cdot\begin{pmatrix} 1 & \text{IF } X=0 \\ 2 & \text{IF } X=1 \end{pmatrix} = 3(1+X)Y$$

$$\mathbb{E}(g(X,Y)) = \sum_{k,l} g(k,l)\, p_{X,Y}(k,l)$$

$$\sum_{k,l} g(k,l)\, p_{X,Y}(k,l) = \sum_{\substack{k\in\{0,1\}\\ l\in\{0,1,2,3\}}} g(k,l)\, p_{X,Y}(k,l)$$

$$= \sum_{k,l} 3(1+k)\,l \; p_{X,Y}(k,l)$$

$$= \underbrace{3\cdot(1+0)\cdot 1\cdot\frac{1}{4}}_{0.75} + \underbrace{3(1+1)\cdot 1\cdot\frac{1}{8}}_{0.25} + \underbrace{3\cdot(1+0)\cdot 2\cdot\frac{1}{8}}_{0.75} + \underbrace{3\cdot(1+1)\cdot 2\cdot\frac{1}{4}}_{3}$$

$$+ \underbrace{3(1+1)\cdot 3\cdot\frac{1}{8}}_{2.25} = 7.5$$

$P(Y=2) = 3/8$

$l=0$

$P(X=0) = 1/2$

X \ Y	0	1	2	3
0	1/8 HHH	HTH, HHT 1/4	HTT 1/8	0
1	0	THH 1/8	THT, TTH 1/4	TTT 1/8

$\mathbb{E}(\text{REWARD}) = \$7.5$

NOTE : EVEN THOUGH $X=0$ & $Y=1$ ARE BOTH POSSIBLE

$$p(0,1) = P(X=0, Y=1) = 0$$

MARGINAL DISTRIBUTIONS

e.g. $P(X=0)$ $\qquad \{X=0\} = \bigcup_{j=0}^{3} \{X=0, Y=j\}$

$$\Rightarrow P(X=0) = \sum_{j=0}^{3} P(X=0, Y=j) = \sum_{j=0}^{3} p(0,j) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} + 0$$

$$= 1/2$$

$P(Y = 2)$ $\{Y=2\} = \{X=0, Y=2\} \cup \{X=1, Y=2\}$

$$\Rightarrow P(Y=2) = P(X=0, Y=2) + P(X=1, Y=2)$$

$$= \sum_{k=0}^{1} p(k, 2) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

UPSHOT: YOU CAN FIND THE p.m.f OF X (RESP. Y) FROM THE JOINT p.m.f.

$$p_X(k) = P(X = k) = \sum_{l} P(X=k, Y=l) = \sum_{l} p_{X,Y}(k, l)$$

Fact 6.4. Let $p(k_1, \ldots, k_n)$ be the joint probability mass function of $(X_1, \ldots, X_n)$. Let $1 \le j \le n$. Then the probability mass function of X_j is given by

$$p_{X_j}(k) = \overset{(n-1)\text{-fold sum}}{\sum_{\ell_1,\ldots,\ell_{j-1},\ell_{j+1},\ldots,\ell_n}} p(\ell_1, \ldots, \ell_{j-1}, k, \ell_{j+1}, \ldots, \ell_n), \tag{6.4}$$

where the sum is over the possible values of the other random variables. The function p_{X_j} is called the *marginal probability mass function* of X_j.

AFTER IGNORING THE RANDOMNES IN $\{X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\}$

Pf: IN THE BOOK. $\{X_j = k\} = \bigcup_{\ell_1,\ldots,\ell_{j-1}} \bigcup_{\ell_{j+1},\ldots,\ell_n} \{X_1 = \ell_1, \ldots, X_{j-1} = \ell_{j-1}, X_j = k, X_{j+1} = \ell_{j+1}, \ldots X_n = \ell_n\}$

For example, for two random variables X and Y we get the formulas

$$p_X(x) = \sum_y p_{X,Y}(x,y) \quad \text{and} \quad p_Y(y) = \sum_x p_{X,Y}(x,y). \tag{6.5}$$

If the joint probability mass function of X and Y is presented as a table, then these are the row and column sums.

$P(Y=0) = \frac{1}{8}$ $\qquad$ $P(Y=2) = \frac{3}{8}$ $\qquad$ $P(Y=3) = 1/8$

		Y			
		0	1	2	3
X	0	$1/8$ HHH	HTH, $1/4$ HHT	HTT $1/8$	0
	1	0	THH $1/8$	THT, $1/4$ TTH	TTT $1/8$

$P(X=0) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8}$
$= 1/2$

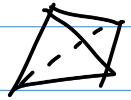
$P(X=1) = \frac{1}{8} + \frac{1}{4} + \frac{1}{8} = \frac{1}{2}$

$P(Y=1) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$

BACK AT

10 : 03 AM

Example 6.6. A tetrahedron shaped die gives one of the numbers 1, 2, 3, 4 with equal probabilities. We roll two of these dice and denote the two outcomes by X_1 and X_2. Let $Y_1 = \min(X_1, X_2)$ and $Y_2 = |X_1 - X_2|$. Find (i) the joint probability mass function of X_1, X_2, (ii) the joint probability mass function of Y_1, Y_2, (iii) the marginal probability mass functions of Y_1 and Y_2, (iv) the probability $P(Y_1 Y_2 \leq 2)$, and (v) $E[Y_1 Y_2]$.



(d4)

$X_1, X_2 \in \{1,2,3,4\}$

$Y_1 = \min\{X_1, X_2\}$

$Y_2 = |X_1 - X_2|$

(i)

$P(X_1 = a, X_2 = b) = \frac{1}{16}$

$$p_{X_1,X_2}(a,b) = \begin{cases} 1/16 & \text{if } 1 \leq a,b \leq 4 \\ 0 & \text{o.w.} \end{cases}$$

(ii)

$$Y_1 = \min(X_1, X_2) \in \{1,2,3,4\}$$

$$Y_2 = |X_1 - X_2| \in \{0, 1, 2, 3\}$$

$$P_{Y_1,Y_2}(k, l) = P(Y_1 = k, Y_2 = l)$$

EITHER $X_1 = k$ OR $X_2 = k$

$X_2 = k + l$ $\qquad$ $X_1 = k + l$

$(|X_2 - X_1| = Y_2 = l)$

$X_2 \geqslant X_1$

$$\ell = 0 \implies |X_1 - X_2| = 0 \implies X_1 = X_2$$

$$P(Y_1 = k, Y_2 = 0) = P(X_1 = X_2 = k)$$

$$= P(X_1 = k, X_2 = k)$$

$$= \frac{1}{16} \quad (\text{PROVIDED } k \in \{1, \dots, 4\})$$

$\ell \neq 0$

EITHER $X_1 = k$ OR $X_2 = k$

$X_2 = k + \ell$ $\qquad$ $X_1 = k + \ell$

$$P(Y_1 = k, Y_2 = \ell) = P(X_1 = k, X_2 = k + \ell) + P(X_2 = k, X_1 = k + \ell)$$

$$P(Y_1 = k, Y_2 = l) = 2P(X_1 = k, X_2 = k + l)$$

$$= \begin{cases} 2 \cdot \dfrac{1}{16} & \text{IF} \quad k,\ k + l \in \{1, \ldots, 4\} \\ 0 & \text{o.w.} \end{cases}$$

$Y_1 \backslash Y_2$	0	1	2	3
1	1/16	1/8	1/8	1/8
2	1/16	1/8	1/8	0
3	1/16	1/8	0	0
4	1/16	0	0	0

p.m.f. TABLE

p_{Y_1, Y_2}

(iii)

MARGINAL p.m.f. OF Y_2.

	$P(Y_2=l)$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{8}$	$\boxed{1}$
$P(Y_1=k)$	$Y_1 \backslash Y_2$	0	1	2	3	
$\frac{7}{16}$	1	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	
$\frac{5}{16}$	2	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	0	
$\frac{3}{16}$	3	$\frac{1}{16}$	$\frac{1}{8}$	0	0	
$\frac{1}{16}$	4	$\frac{1}{16}$	0	0	0	
$\boxed{1}$						

MARGINAL p.m.f. OF Y_1

$p_{X,Y}(k,l)$

(iv) $P(Y_1 Y_2 \leq 2)$

$$\{Y_1 Y_2 \leq 2\} = \{Y_2 = 0\} \cup \{Y_1 = 1, \; Y_1 \leq 2\} \cup \{Y_1 = 2, \; Y_2 = 1\}$$

$Y_1 \backslash Y_2$	0	1	2	3
1	1/16	1/8	1/8	1/8
2	1/16	1/8	1/8	0
3	1/16	1/8	0	0
4	1/16	0	0	0

$$P(Y_1 Y_2 \leq 2) = P(Y_1 = 0) + P(Y_1 = 1, Y_2 = 1) + P(Y_1 = 1, Y_2 = 1)$$

$$+ P(Y_2 = 2, Y_1 = 1) = \frac{1}{4} + \left(\frac{1}{8} + \frac{1}{8}\right) + \frac{1}{8} = 5/8$$

(v)

$$\mathbb{E}(Y_1 Y_2) = \sum_{k,l} \underbrace{kl}_{Y_1 Y_2 \; (Y_1 = k,\, Y_2 = l)} \; p_{Y_1,Y_2}(k,l)$$

$Y_1 \backslash Y_2$	0	1	2	3
1	1/16	$\frac{1}{8} \cdot 1$	$\frac{1}{8} \cdot 2$	$\frac{1}{8} \cdot 3$
2	1/16	$\frac{1}{8} \cdot 2$	$\frac{1}{8} \cdot 4$	0
3	1/16	$\frac{1}{8} \cdot 3$	0	0
4	1/16	0	0	0

FACTS: ① $l = 0$ DOES NOT CONTRIBUTE

② $p(\cdot,\cdot) = 0$ DOESN'T CONTR.

$p_{Y_1,Y_2}(k,l) = \frac{1}{8}$, $\mathbb{E}(Y_1 Y_2) = \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 2 + \dots + 3 \cdot \frac{1}{8}$

$= 15/8$

BACK AT

10:25 AM

INTERLUDE : REVIEW OF DOUBLE & MULTIPLE INTEGRALS

$$\left.\begin{aligned} f : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ z &= f(x,y) \end{aligned}\right\}$$ FUNCTION OF 2 VARIABLES.

(PARABOLOID)

e.g. $z = x^2 + y^2$

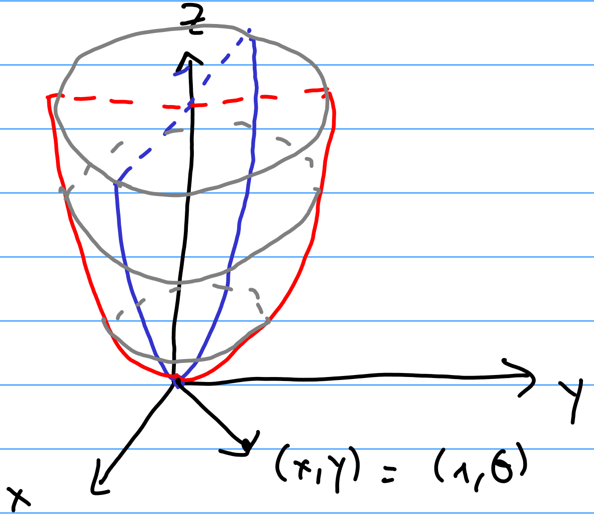
$(x,y) = (0,0) \Rightarrow z = 0$

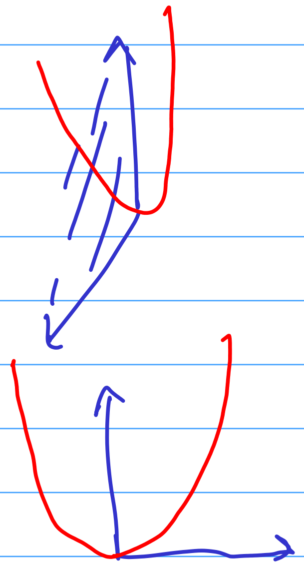
$y = 0, \quad z = x^2$

$x = 0, \quad z = y^2$

$r^2 = x^2 + y^2$

$z = r^2$

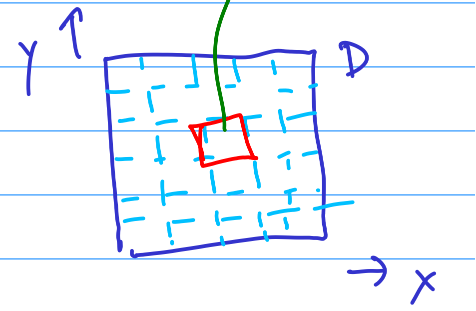


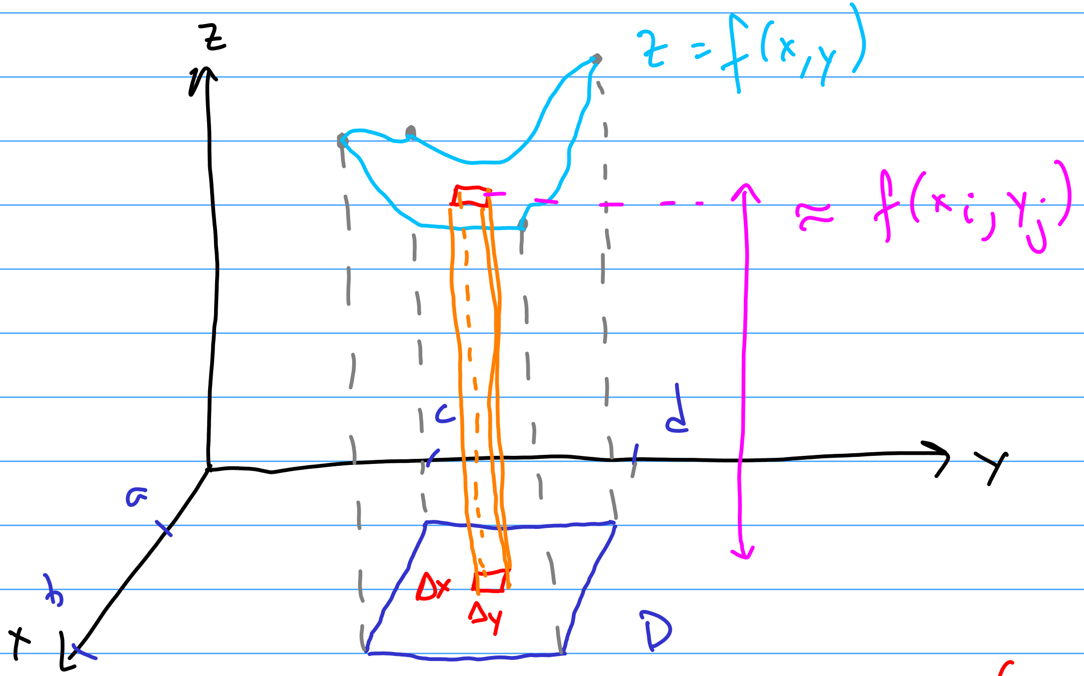


(ASSUMING f IS CONT.)

WE WANT TO FIND (SIGNED) VOLUME UNDER THE SURFACE $z = f(x,y)$ OVER A (RECTANGULAR) REGION $D = [a,b] \times [c,d]$

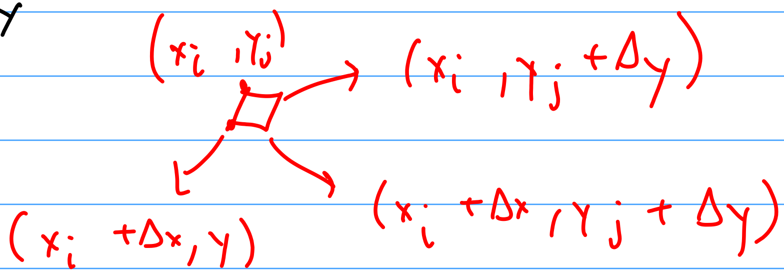
$\Delta A = \Delta x \cdot \Delta y$





VOLUME OF "FRENCH FRY" = LEN × WID × HGT.

$\approx f(x_i, y_j)\, \Delta x\, \Delta y$



VOL. OF (i,j)th FRY $\approx$ $f(x_i, y_j)\,\Delta x\,\Delta y$

$\therefore$ TOTAL VOL. $\approx \sum_{i,j} f(x_i, y_j)\,\Delta x\,\Delta y$

RIEMANN SUM

AS $\Delta x, \Delta y \longrightarrow 0$, # OF SUMMANDS GOES TO ∞.

IN YOUR $dA = dx\,dy$

$\therefore$ WE SAY

$$\iint_D f(x,y)\,dA = \lim_{\Delta x, \Delta y \to 0} \sum_{i,j} f(x_i, y_j)\,\Delta x\,\Delta y$$

Why ONE DEFINES FOR

$f : \mathbb{R}^n \longrightarrow \mathbb{R} \longrightarrow y = f(x_1, x_2, \ldots, x_n)$

$R = \prod_{j=1}^{n} [a_j, b_j] = \{ (x_1, \ldots, x_n) : \text{FOR ALL } j, \ x_j \in [a_j, b_j] \}$

$\underbrace{n\text{-fold}}$ integral

$$\underset{R}{\int \cdots \int} f(x_1, x_2, \ldots, x_n) \underbrace{dx_1 \cdots dx_n}_{\text{GENERALIZED VOLUME}} = \lim_{\Delta x_j \to 0} \sum_{i_1, i_2, \ldots, i_n} f(x_1, \ldots, x_n) \Delta x_1 \cdots \Delta x_n$$

HOW TO COMPUTE W/ THIS DEFN?

FUBINI'S THEOREM

IF f IS CONT. ON $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \overbrace{\left(\int_{y=c}^{y=d} f(x,y)\, dy \right)}^{\text{INTEGRATE } y\text{, PRETENDING THAT } x \text{ IS CONSTANT}} dx$$

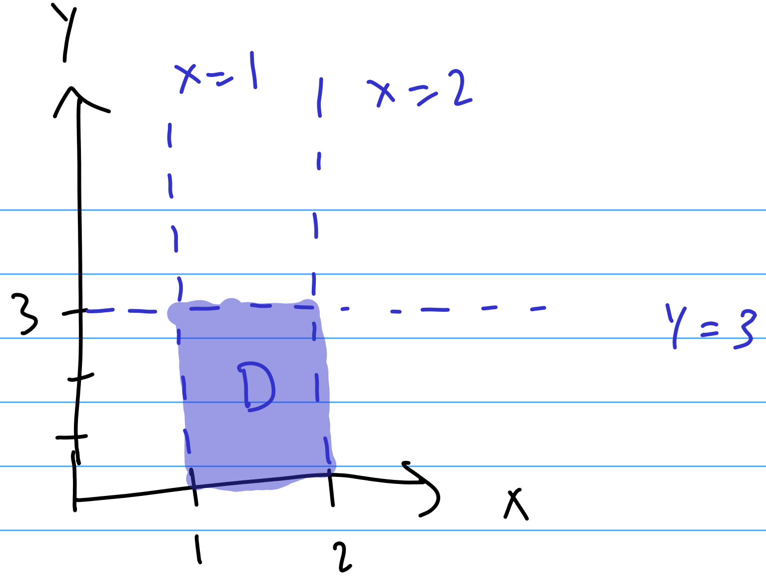
INT. OVER x AFTER

1 VAR. INTEGRALS.

$$= \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\, dx \right) dy$$

ITERATED INTEGRALS.

e.g. $D = [1,2] \times [0,3]$



$$\iint_D (x^2 y + y^2)\, dA$$

(Y- FIRST)

$$= \int_{x=1}^{x=2} \left(\int_{Y=0}^{Y=3} (x^2 y + y^2)\, dy \right) dx = \int_{x=1}^{x=2} \left[\frac{x^2 y^2}{2} + \frac{y^3}{3} \right]_{Y=0}^{Y=3} dx$$

$$= \int_{x=1}^{x=2} \left(\frac{9x^2}{2} + 9 \right) dx = \left[\frac{3x^3}{2} + 9x \right]_{x=1}^{x=2} = \frac{3}{2} \cdot 2^3 + 9 \cdot 2 - \frac{3}{2} - 9$$

$$= \boxed{19.5}$$

X-FIRST

$$\iint_D (x^2y + y^2)\,dA = \int_{y=0}^{y=3} \left[\int_{x=1}^{x=2} (x^2y + y^2)\,dx \right] dy$$

$$= \int_{y=0}^{y=3} \left[\frac{x^3y}{3} + xy^2 \right]_{x=1}^{x=2} dy = \int_{y=0}^{y=3} \left[\frac{8}{3}y + 2y^2 - \frac{y}{3} - y^2 \right] dy$$

$$= \int_0^3 \left(\frac{7y}{3} + y^2 \right) dy = \frac{7y^2}{6} + \frac{y^3}{3} \Big]_{y=0}^{y=3} = \frac{7 \cdot 3^2}{6} + \frac{3^3}{3}$$

$$= \frac{21}{2} + 9 = \boxed{19.5}$$

FUBINI'S THEOREM HOLDS ALSO IN HIGHER DIMENSIONS.

e.g.

$$\underbrace{\iiint_{(x,y,z)\in[0,1]^3} xyz \; dx\,dy\,dz}_{= \, 1/8} = \int_0^1 \left[\int_0^1 \left(\int_0^1 xyz \, dx \right) dy \right] dz$$

$$= \int_0^1 \left[\int_0^1 \left(\frac{yz}{2} \right) dy \right] dz = \int_0^1 \frac{z}{4} \, dz$$

$$= \frac{1}{8}$$

$$\overbrace{\iiiint_{0 \le x_j \le 1} 5 \, dx_1 \, dx_2 \, dx_3 dx_4}^{5\,.}$$

$$(x_1, x_2, x_3 x_4) \in [0,1]^4$$

$$= \int_{x_4=0}^{x_4=1} \left[\int_{x_2=0}^{x_2=1} \left[\int_{x_1=0}^{x_1=1} \left[\int_{x_3=0}^{x_3=1} 5 \, dx_3 \right] dx_1 \right] dx_2 \right] dx_4$$

$$\boxed{\int_{x=0}^{x=1} 5 \, dx = 5x \Big]_0^1 = 5}$$

$$= \int_{x_4=0}^{x_4=1} \left[\int_{x_2=0}^{x_2=1} \left[\int_{x_1=0}^{x_1=1} 5 \, dx_1 \right] dx_2 \right] dx_4$$

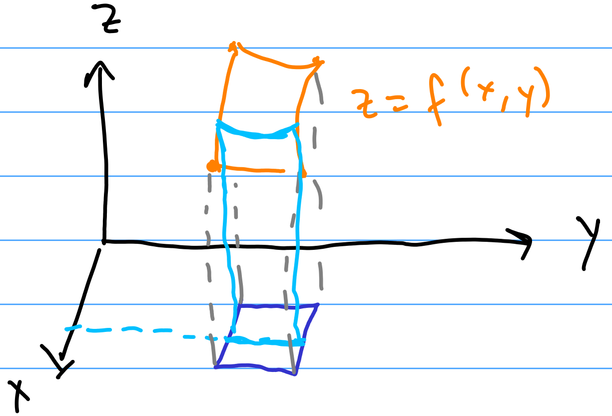
$$= \int_{x_4=0}^{x_4=1} \left[\int_{x_2=0}^{x_2=1} 5 \, dx_2 \right] dx_4 = \int_{x_4=0}^{x_4=1} 5 \, dx_4 = 5$$

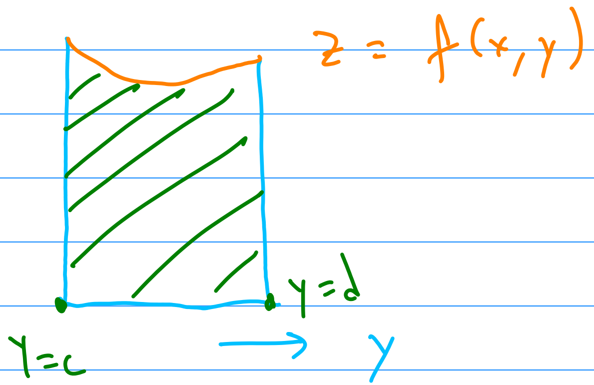
WHAT IF DOMAIN OF INTEGR. IS NOT RECTANGULAR?

IF $R = [a,b] \times [c,d]$

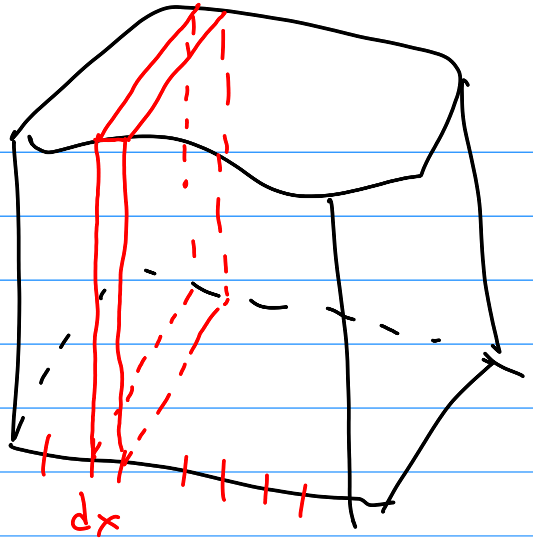
$A(x) \rightarrow$ CROSS-SECTIONAL AREA

$$\iint_R f(x,y)\,dA = \int_{x=a}^{x=b} \overbrace{\left[\int_{y=c}^{y=d} f(x,y)\,dy\right]}^{A(x)} dx$$





(x - FIXED)



CROSS-SECTIONAL AREA $\approx A(x)$

VOL $\simeq A(x) \cdot dx$

D → NOT RECTANGULAR

$$\iint_D f(x,y)\, dA = \int_a^b \underbrace{A_D(x)}_{\text{CROSS-SECTIONAL AREA.}}\, dx$$