MATH 201 (SUMMER 2023, SESH A2)

LECTURE 12: 06/06/23

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. REMINDER: MIDTERM REGRADES WILL CLOSE ON WED, JUNE 7th AT 11 PM ET

2. OFFICE HOURS: TR: 11:15 AM - 12:15 PM
   i.e. INCLUDES TODAY, AFTER CLASS.

3. UPCOMING DEADLINES:
   a. HW 6 - WED
   b. WW 6 - WED
   TOMORROW!
   c. WW 7 - SAT
   d. HW 7 - SUN

4. HW02 & HW03 IS GRADED.

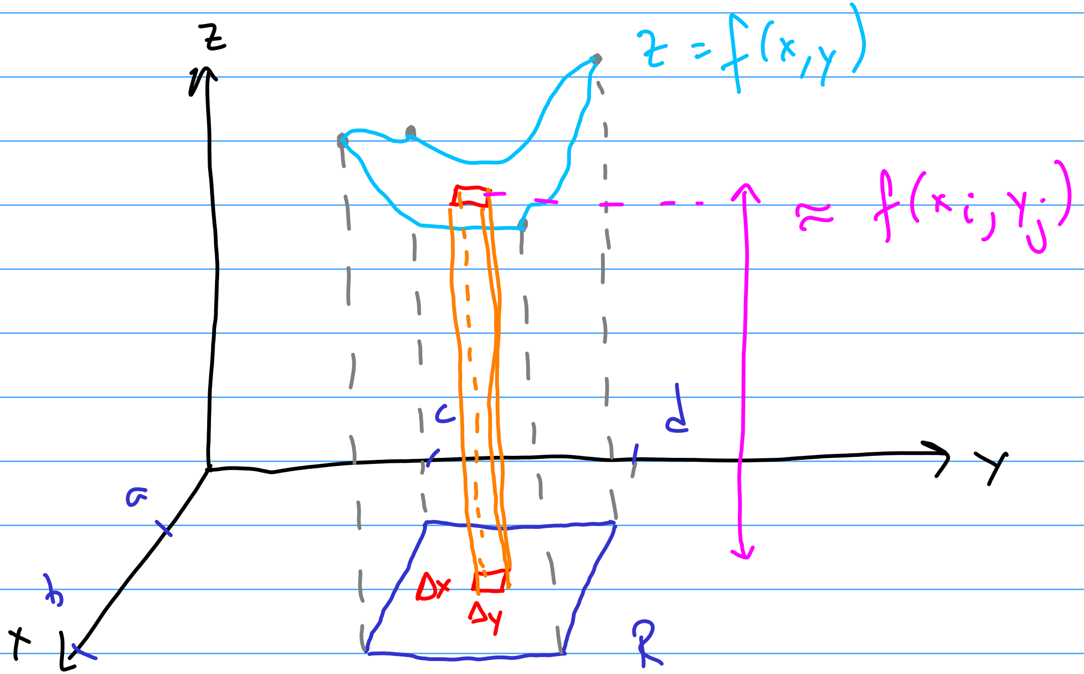
ANNOUNCEMENTS

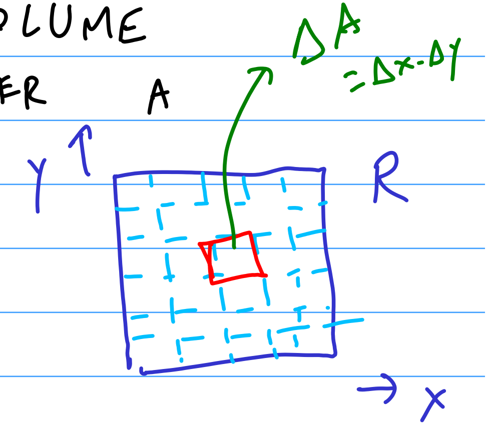
5. PLEASE FILL OUT MID-SEM FEEDBACK. → ANONYMOUS → OPTIONAL
6. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

RECALL

(ASSUMING f IS CONT.)

WE WANT TO FIND (SIGNED) VOLUME UNDER THE SURFACE $z = f(x,y)$ OVER A (RECTANGULAR) REGION $R = [a,b] \times [c,d]$





VOLUME OF "FRENCH FRY" = LEN × WID × HGT. $\approx f(x_i, y_j)\,\Delta x\,\Delta y$

$$\boxed{\iint_R f(x,y)\,dA}$$

RECALL

Illy ONE DEFINES FOR

$$f : \mathbb{R}^n \longrightarrow \mathbb{R} \quad \longrightarrow \quad y = f(x_1, x_2, \ldots, x_n)$$

$$R = \prod_{j=1}^{n} [a_j, b_j] = \{ (x_1, \ldots, x_n) : \text{FOR ALL } j, \ x_j \in [a_j, b_j] \}$$

n-fold integral

$$\underbrace{\int \cdots \int}_{R} f(x_1, x_2, \ldots, x_n) \underbrace{dx_1 \ldots dx_n}_{\text{GENERALIZED VOLUME}} = \lim_{\Delta x_j \to 0} \sum_{i_1, i_2, \ldots i_n} f(x_1, \ldots, x_n) \Delta x_1 \ldots \Delta x_n$$

RECALL

FUBINI'S THEOREM

IF f IS CONT. ON $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\,dA = \underbrace{\int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x,y)\,dy \right) dx = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x,y)\,dx \right) dy}_{\text{ITERATED INTEGRALS.}}$$

SOMETHING SIMILAR HOLDS IN HIGHER DIMENSIONS TOO (cf. LECTURE 11).

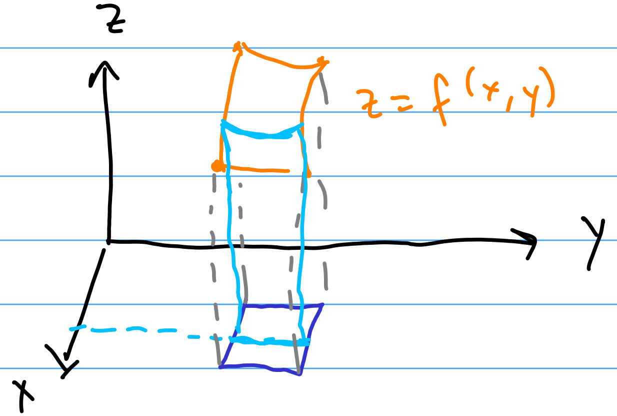
RECALL

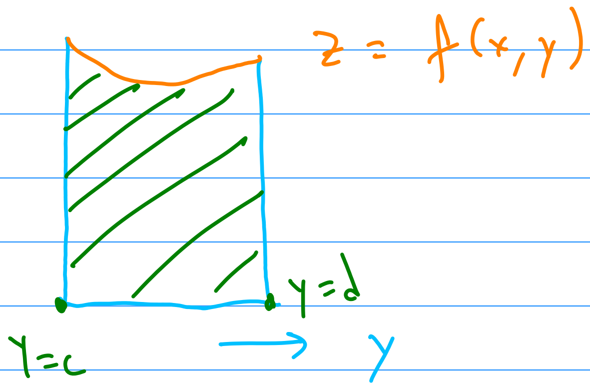
WHAT IF DOMAIN OF INTEGR. IS NOT RECTANGULAR?

IF $R = [a,b] \times [c,d]$

$$\iint_R f(x,y)\, dA = \int_{x=a}^{x=b} \underbrace{\left[\int_{y=c}^{y=d} f(x,y)\, dy \right]}_{}\, dx$$

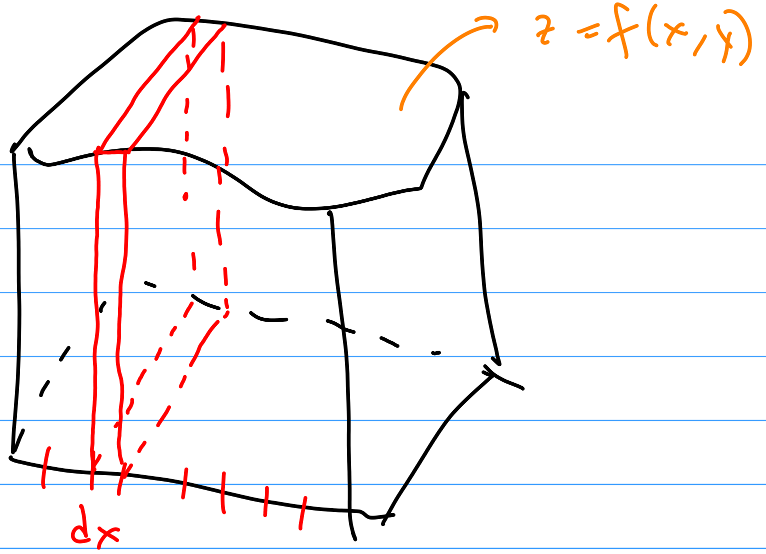
$A(x) \to$ CROSS-SECTIONAL AREA





(x - FIXED)

RECALL



CROSS-SECTIONAL AREA $\approx A(x)$

VOL $\approx A(x) \cdot dx$

$D \to$ NOT RECTANGULAR

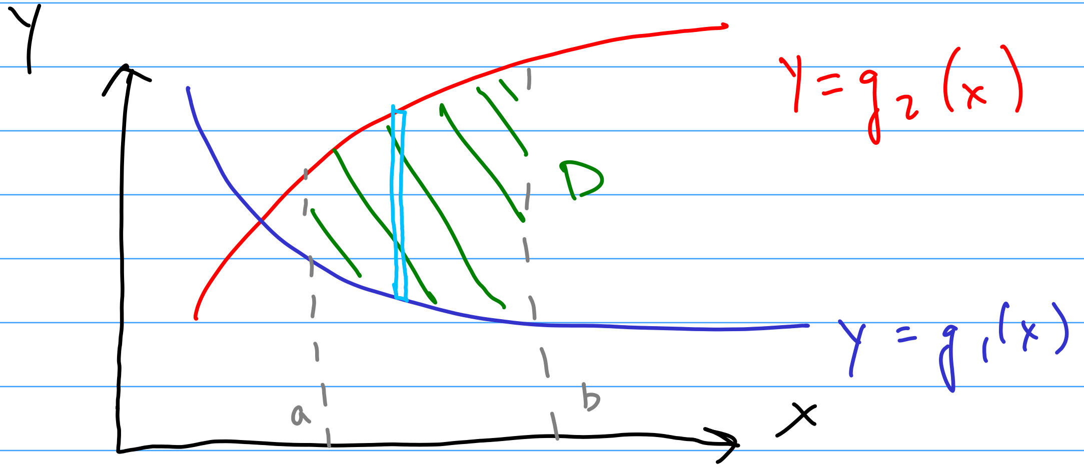
$$\iint_D f(x,y)\, dA = \int_a^b \underbrace{A_D(x)}_{\text{CROSS-SECTIONAL AREA. OF THE CUT AT } x.}\, dx$$

THINK OF : WASHER / DISC METHOD FROM CALCULUS.

GENERAL DOUBLE INTEGRALS

TYPE I :

$$D = \left\{ (x,y) : \begin{array}{c} a \le x \le b \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$



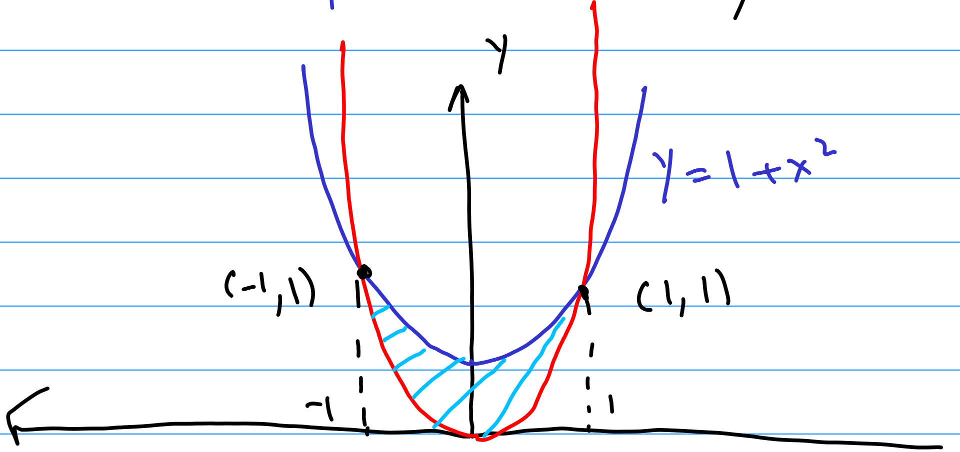
g_j ARE CONT.

FOR TYPE I :

$$\iint_D f(x,y)\,dA = \int_{x=a}^{x=b} \underbrace{\left[\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy \right]}_{\text{CROSS-SECTIONAL AREA}} dx$$

e.g. $$\iint_D (x+2y)\,dA = \int_{-1}^{1}\left[\int_{y=2x^2}^{y=x^2+1}(x+2y)\,dy\right]dx$$

D $\rightsquigarrow$ REGION BOUNDED BY $\overbrace{y=1+x^2}^{g_1}$, $\underbrace{y=2x^2}_{g_2}$



$$1+x^2 = 2x^2$$
$$\Rightarrow x^2 = 1$$
$$\Rightarrow x = \pm 1$$

$$\int_{x=-1}^{x=1}\left[\int_{y=2x^2}^{y=x^2+1}(x+2y)\,dy\right]dx = \int_{x=-1}^{x=1}\left[xy+y^2\right]_{y=2x^2}^{y=x^2+1}dx$$

$$= \int_{-1}^{1}\left[x(x^2+1)+(x^2+1)^2-x(2x^2)-(2x^2)^2\right]dx$$

$$= \int_{-1}^{1}\left(-3x^4-x^3+2x^2+x+1\right)dx$$

TYPE II:

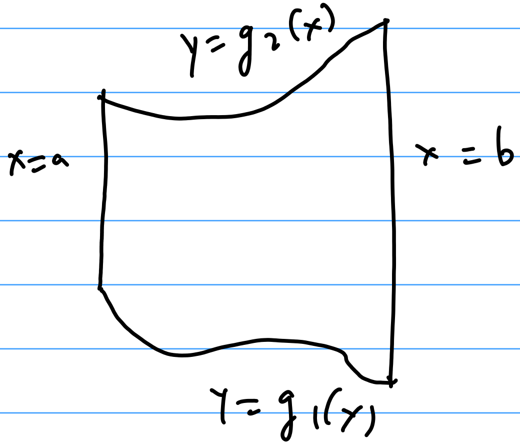
$$D = \left\{ (x,y) \ : \ \begin{matrix} c \le y \le d \\ h_1(y) \le x \le h_2(y) \end{matrix} \right\}$$

[IDEA: SWAP x & y IN TYPE 1.]

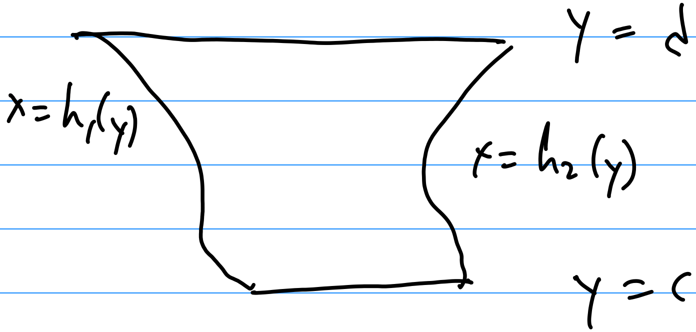
IN THIS CASE,

$$\iint_D f(x,y)\, dA = \int_{y=c}^{y=d} \left[\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\, dx \right] dy$$

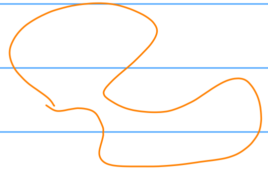
TYPE I.



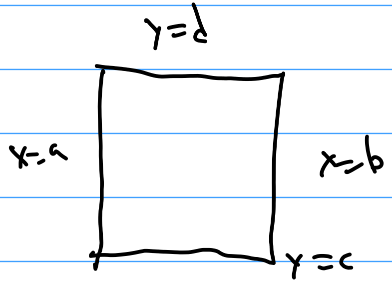
TYPE II



NOT EXHAUSTIVE



NOT MUTUALLY EXCLUSIVE



NOTE: TYPE I & TYPE II ARE NOT MUTUALLY EXCLUSIVE.

THIS CAN HELP SIMPLIFY SOME ITERATED INTEGRALS.

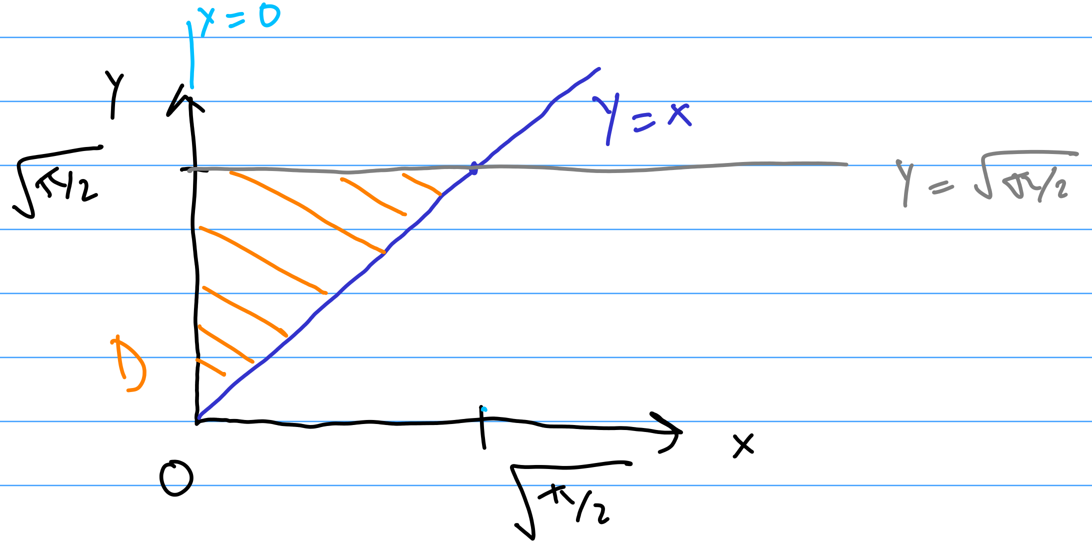
e.g. $\underbrace{\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \sin(y^2)\, dy\, dx}_{\text{ITERATED INTEGRAL}}$

$\sin(y^2)$ NOT INTEGRABLE IN ELEMENTARY

(TYPE I)

APPLY FUBINI IN REVERSE.

$$\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA$$



THIS IS ALSO A TYPE II DOMAIN.

$c = 0$, $d = \sqrt{\pi/2}$

$0 \le x \le y$

$h_1(y)$ $h_2(y)$

FUBINI SAYS:

$$\int_0^{\sqrt{\pi/2}}\int_x^{\sqrt{\pi/2}} \sin(y^2)\,dy\,dx = \iint_D \sin(y^2)\,dA = \int_0^{\sqrt{\pi/2}}\int_0^{y} \sin(y^2)\,dx\,dy$$

$$= \int_0^{\sqrt{\pi/2}} \Big[\, x \sin(y^2) \,\Big]_{x=0}^{x=y} dy = \int_0^{\sqrt{\pi/2}} y\,\underbrace{\sin(y^2)}_{\sin u}\,dy$$

$\frac{du}{2}$

$u = y^2 \Rightarrow du = 2y\,dy$

$$\int_0^{\pi/2} \sin u \, du = -\cos u \Big]_0^{\pi/2} = \cancel{-\cos\frac{\pi}{2}} + \cos \overset{1}{\cancel{0}}$$

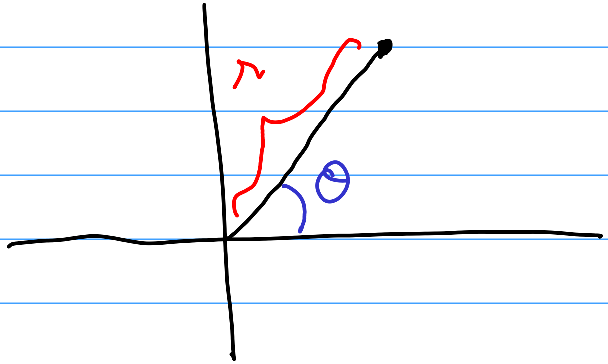
$$= 1$$

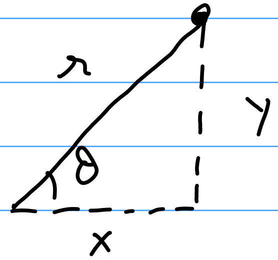
$$\boxed{\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \sin(y^2)\, dy\, dx = 1}$$

POLAR COORDINATES

IN TWO DIMENSIONS, WE HAVE POLAR COORDINATES

(r, θ)





$x = r \cos\theta$

$y = r \sin\theta$

HOW TO USE THIS TO COMPUTE DOUBLE INTEGRALS?

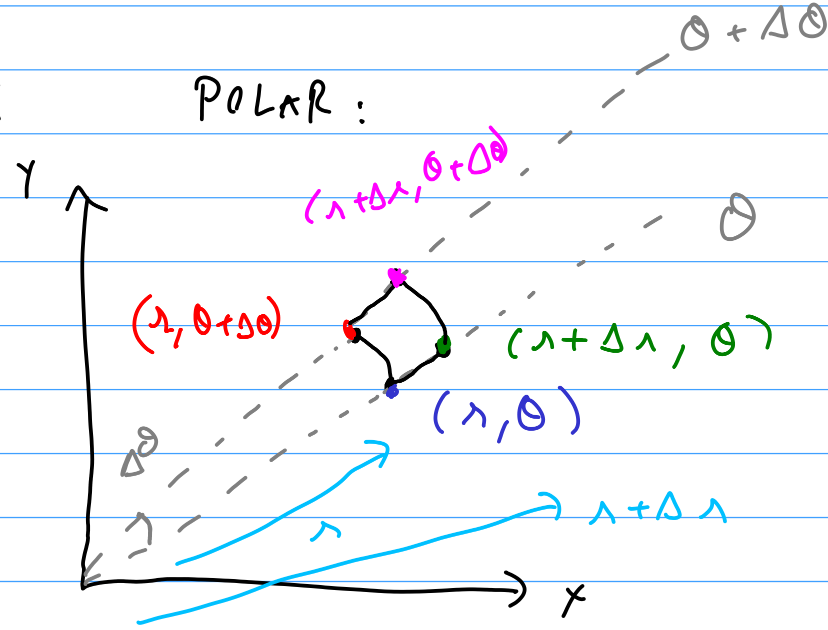
$$dA = dx\,dy = r\,dr\,d\theta$$

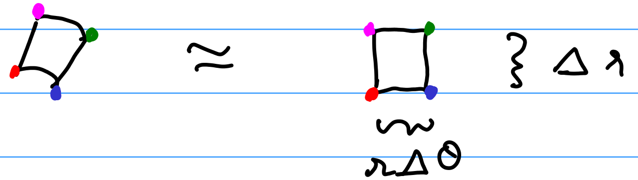
AREA ELEMENT:

$$\Delta A = \Delta x \cdot \Delta y$$

$$\Delta A = r \cdot \Delta r \cdot \Delta \theta$$

IN POLAR:





⌒ → LENGHT = ANGLE × RADIUS

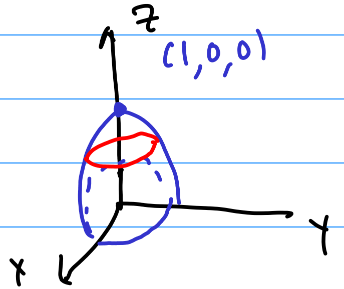
$$\therefore \iint_D f(x,y)\, \underbrace{dx\,dy}_{dA} = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

↖ IN TERMS OF (r,θ)

e.g.

FIND THE VOLUME OF SOLID BOUNDED BY

$z = 0$ & $z = 1 - x^2 - y^2$.

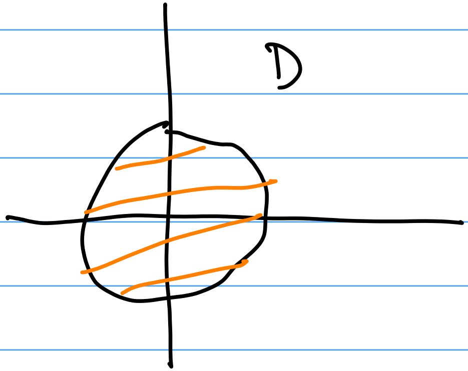


$0 = 1 - x^2 - y^2$

$(z=1, \quad x^2 + y^2 = 0$

$D = \{ (x, y) : x^2 + y^2 \leq 1 \}$

TYPE I, II.



$r^2 = x^2 + y^2$

$$D = \left\{ (r, \theta) : \begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq \theta < 2\pi \end{array} \right\}$$

$$\text{VOL.} = \iint_D f(x,y)\, dA = \iint_D (1 - \underbrace{x^2 - y^2}_{r^2})\, dA$$

$$= \iint_D \left[1 - \underbrace{(r\cos\theta)^2 - (r\sin\theta)^2}_{r^2 = x^2 + y^2}\right] r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} (1 - r^2)\, r\,dr\,d\theta$$

$$= \int_0^1 \int_0^{2\pi} (1 - r^2)\, r\,d\theta\,dr$$

$$\int_0^1 \int_0^{2\pi} (1-r^2)\, r \, d\theta \, d\theta = \int_0^1 \Big[\theta \cdot (1-r^2)\cdot r\Big]_{\theta=0}^{\theta=2\pi} dr$$

$$= 2\pi \int_0^1 (1-r^2)\cdot r \, dr$$

$$= 2\pi \left[\frac{r^2}{2} - \frac{r^4}{4}\right]_{r=0}^{r=1}$$

$$= 2\pi \left[\frac{1}{2} - \frac{1}{4}\right] = 2\pi \cdot \frac{1}{4} = \frac{\pi}{2}$$

WE CAN NOW SHOW

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = 1$$

→ NEEDED FOR NORMAL p.d.f.

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} \, dx \quad — \quad \mathrm{I}$$

1 IN I REPLACE x BY y :

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \, dy \quad — \quad \mathrm{II}$$

② MULTIPLY Ⓘ BY Ⓘ

$$I^2 = I \cdot I = \left(\frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-x^2/2}\, dx}_{\text{CONSTANT W.R.T. } y} \right) \cdot \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \right)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right) \underbrace{e^{-y^2/2}}_{\text{CONSTANT W.R.T. } x} dy$$

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \underbrace{\left(e^{-x^2/2}\right) \cdot \left(e^{-y^2/2}\right)}_{e^{-x^2/2} \cdot e^{-y^2/2} = e^{\frac{-x^2-y^2}{2}} = e^{-\frac{(x^2+y^2)}{2}}} dx \right] dy$$

③ SUMMARY:

$$I^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2} \, dx \, dy$$

④ DOUBLE INTEGRAL:

$$I^2 = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-\overbrace{(x^2+y^2)}^{r^2}/2} \, dA$$

5. CONVERT TO POLAR:

$$x^2 + y^2 = r^2$$

$$\mathbb{R}^2 = \{ (x, y) : x \in (-\infty, \infty), y \in (-\infty, \infty) \}$$

$$= \left\{ (r, \theta) : \begin{array}{l} 0 \le r < \infty \\ 0 \le \theta < 2\pi \end{array} \right\}$$

$$dA = r \, d\theta \, dr$$

$$I^2 = \frac{1}{2\pi} \iint_{\mathbb{R}^2} e^{-r^2/2} (r \, d\theta \, dr) = \frac{1}{2\pi} \int_{r=0}^{r=\infty} \int_{\theta=0}^{2\pi} r e^{-r^2/2} \, d\theta \, dr$$

(6)

$$\frac{1}{2\pi}\int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi} r\, e^{-r^2/2}\, d\theta\, dr$$

$$= \int_0^\infty \boxed{r\, e^{-r^2/2}}\, dr$$

$$\frac{d}{dr}\left[-e^{-r^2/2}\right]$$

$$= -e^{-r^2/2}\Big]_0^\infty = \lim_{r\to\infty}\left[\cancel{-e^{-r^2/2}} + e^{0^2/2}\right]$$

$$= 1$$

$$I^2 = 1 \Rightarrow I = 1 \quad (I \geq 0)$$

Q.E.D.

BREAK TILL

10:20 AM

§ 6.2 JOINT DENSITY FOR CONTINUOUS R.V.s

CONTRAST W/ DISCRETE

Definition 6.12. Random variables $X_1, \ldots, X_n$ are **jointly continuous** if there exists a **joint density function** f on $\mathbb{R}^n$ such that for subsets $B \subseteq \mathbb{R}^n$, ♣

$$\underbrace{P((X_1, \ldots, X_n) \in B)}_{\text{JOINT DISTRIBUTION}} = \int \cdots \int_B f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n. \tag{6.9}$$

COMPARE

$$P(X \in B) = \int_B f(x)\, dx$$

REMARK: X_1 & X_2 CONT. ON Ω

$\not\Rightarrow$ X_1 & X_2 ARE JOINTLY CONT.

$\Leftarrow$

AS WITH DISCRETE CASE & ONE-DIM. CASE,

1. $f(x_1, \dots, x_n) \geq 0$

2. $$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \dots, x_n)\, dx_1 \cdots dx_n = 1.$$

||| TO $\sum p(x_1, \dots, x_n) = 1$, $\int_{\mathbb{R}} f(x)\, dx = 1$

3. **Fact 6.13.** Let $g : \mathbb{R}^n \to \mathbb{R}$ be a real-valued function of an n-vector. If $X_1, \dots, X_n$ are random variables with joint density function f then

$$E[g(X_1, \dots, X_n)] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, \dots, x_n) f(x_1, \dots, x_n)\, dx_1 \cdots dx_n \qquad (6.11)$$

provided the integral is well defined.

COMPARE $\sum_{x_1, \dots, x_n} g(x_1, \dots, x_n)\, p(x_1, \dots, x_n)$

$\int_{-\infty}^{\infty} g(x) f(x)\, dx$

Example 6.14. To illustrate calculations with a joint density function, suppose X, Y have joint density function $3xy^2 + y$

$$f(x,y) = \begin{cases} \text{[redacted]}, & 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

① CHECK p.d.f. $\quad f(x,y) \geq 0$ (CLEAR)

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = \int_{y=0}^{y=1} \int_{x=0}^{x=1} \left(3xy^2 + y\right) dx\,dy$$

$$= \int_{y=0}^{y=1} \left[\frac{3x^2y^2}{2} + xy \right]_{x=0}^{y=1} dy = \int_0^1 \left(\frac{3y^2}{2} + y \right) dy$$

$$\int_0^1 \left(\frac{3y^2}{2} + y \right) dy = \frac{y^3}{2} + \frac{y^2}{2} \Big]_{y=0}^{y=1} = \frac{1}{2} + \frac{1}{2} = 1$$

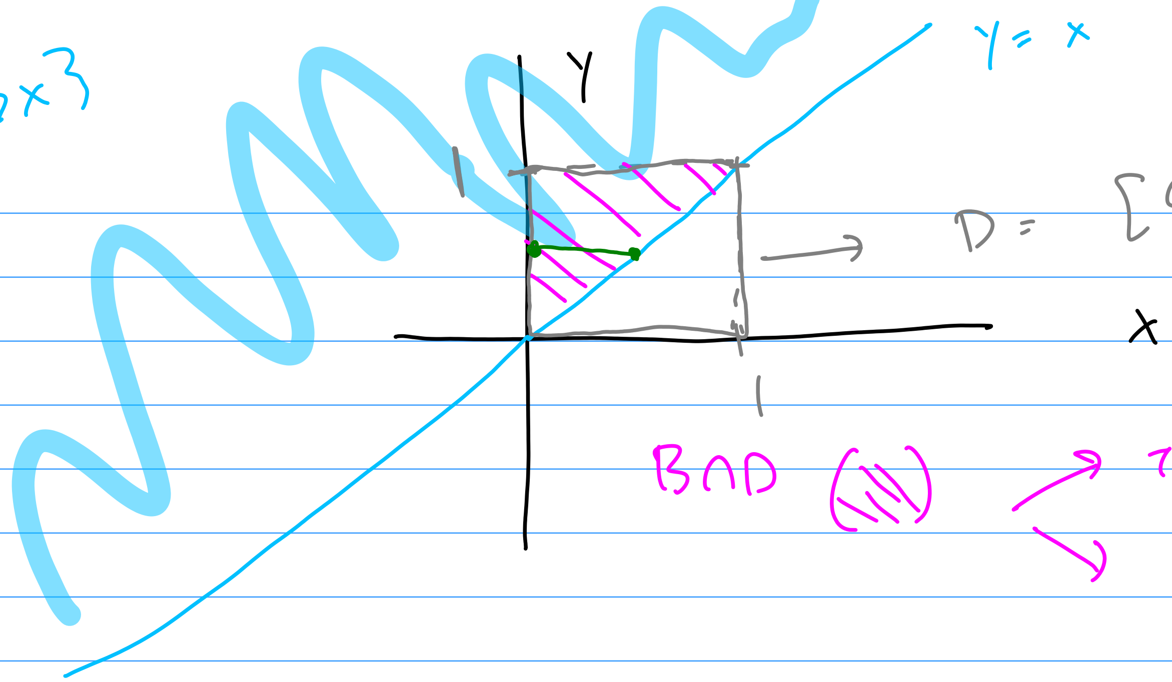
$$\therefore \iint_{\mathbb{R}^2} f(x,y)\, dx\, dy = 1$$

② $P(X < Y) = P\left((X,Y) \in B \right)$

$$B = \{ (x,y) : x < y \}$$

$$P(X < Y) = \iint_B f(x,y)\, dx\, dy$$

$B = \{y > x^3\}$



$D = [0,1]^2$

$B \cap D$

TYPE I

TYPE II

$$\iint_B f(x,y)\,dx\,dy = \iint_{B\cap D} \left(3xy^2 + y\right) dx\,dy$$

TYPE II :

$$\int_{y=0}^{y=1} \int_{x=0}^{x=y} \left(3xy^2 + y\right) dx\, dy$$

$$= \int_{y=0}^{y=1} \left[\frac{3x^2y^2}{2} + xy \right]_{x=0}^{x=y} dy$$

$$= \int_{y=0}^{y=1} \left(\frac{3y^4}{2} + y^2 \right) dy = \left[\frac{3y^5}{10} + \frac{y^3}{3} \right]_{y=0}^{y=1}$$

$$P(X<Y) = \left[\frac{3y^5}{10} + \frac{y^3}{3}\right]_{y=0}^{y=1} = \frac{3}{10} + \frac{1}{3} = \frac{19}{30}$$

$$\mathbb{E}(X^2Y) = \mathbb{E}(g(X,Y)) \qquad g(x,y) = x^2y$$

FACT 3.

$$\mathbb{E}(X^2Y) = \iint_{\mathbb{R}^2} x^2y \, f(x,y) \, dx\,dy$$

$$\mathbb{E}(x^2 y) = \iint_{[0,1]^2} x^2 y \left(3xy^2 + y\right) dx\, dy$$

$$= \int_{x=0}^{x=1} \left[\int_{y=0}^{y=1} \left(3x^3 y^3 + x^2 y^2\right) \right] dx\, dy$$

$$= \int_{x=0}^{x=1} \left[\frac{3x^3 y^4}{4} + \frac{x^2 y^3}{3} \right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1} \left[\frac{3x^3y^4}{4} + \frac{x^2y^3}{3} \right]_{y=0}^{y=1} dx$$

$$= \int_{x=0}^{x=1} \left(\frac{3x^3}{4} + \frac{x^2}{3} \right) dx$$

$$= \left[\frac{3x^4}{16} + \frac{x^3}{9} \right]_{x=0}^{x=1} = \frac{3}{16} + \frac{1}{9}$$

MARGINAL

Fact 6.15. Let f be the joint density function of $X_1, \ldots, X_n$. Then each random variable X_j has a density function f_{X_j} that can be obtained by integrating away the other variables from f:

$$f_{X_j}(x) = \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n-1 \text{ integrals}} f(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_n)\, dx_1 \ldots dx_{j-1}\, dx_{j+1} \ldots dx_n. \tag{6.12}$$

The formula says that to compute $f_{X_j}(x)$, place x in the jth coordinate inside f, and then integrate away the other $n-1$ variables.

For two random variables X and Y the formula is

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy. \tag{6.13}$$

COMPARE

JOINT p.m.f.

$$p_{X_j}(k) = \sum_{\ell_1, \ldots, \ell_{j-1}} \sum_{\ell_{j+1}, \ldots, \ell_n} p(\ell_1, \ldots, \ell_{j-1}, k, \ell_{j+1}, \ldots, \ell_n)$$

Example 6.16. To illustrate Fact 6.15, let us find the marginal density function of X in Example 6.14.

$$f_{X,Y}(x,y) = \begin{cases} 3xy^2 + y & (x,y) \in [0,1]^2 \\ 0 & \text{o.w.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$

NOTE: IF $x \notin [0,1]$, INTEGRAND $\equiv 0$

$\Rightarrow f_X(x) = 0$

$\therefore$ WE ASSUME $x \in [0,1]$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy \qquad (y \in [0,1])$$

$$= \int_{y=0}^{y=1} (3xy^2 + y)\, dy$$

$$= \left[xy^3 + \frac{y^2}{2} \right]_{y=0}^{y=1} = x + 1/2$$

$$\therefore \quad f_X(x) = \begin{cases} x + \frac{1}{2} & \text{IF } 0 \le x \le 1 \\ 0 & \text{o.w.} \end{cases}$$

① $f_X(x) \geq 0$

② $\displaystyle\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^1 \left(x + \frac{1}{2}\right) = \left.\frac{x^2}{2} + \frac{x}{2}\right]_{x=0}^{x=1} = 1$

Example 6.17. Let the joint density function of the random variables X and Y be (see Figure 6.1)

$$f(x,y) = \begin{cases} 2xe^{x^2-y}, & \text{if } 0 < x < 1 \text{ and } y > x^2 \\ 0, & \text{else.} \end{cases} \tag{6.14}$$

Find the marginal density functions f_X of X and f_Y of Y, and compute the probability $P(Y < 3X^2)$.

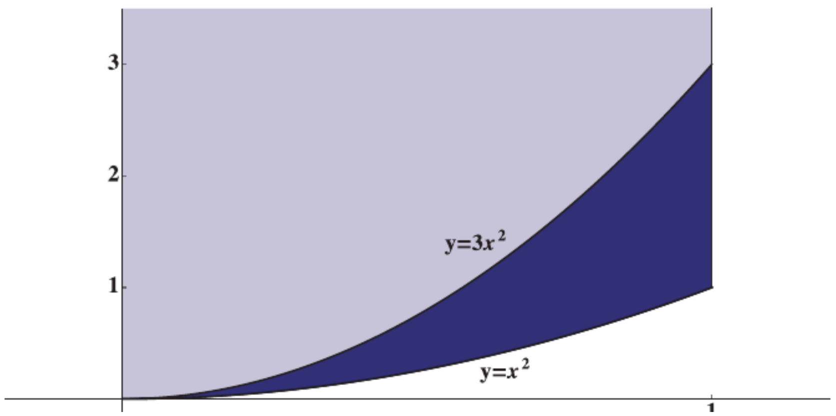


Figure 6.1. The relevant regions in Example 6.17. The dark blue region is the set of points (x, y) satisfying $0 \le x \le 1$ and $x^2 \le y \le 3x^2$.

(1) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

NOTE: $x \geq 1$ OR $x \leq 0$ THEN $f_X(x) = 0$

ASSUME $x \in (0,1)$

$$f_X(x) = \left(\int_{x^2}^{\infty} + \cancel{\int_{-\infty}^{x^2}} \right) = \int_{x^2}^{\infty} 2x e^{x^2 - y}\,dy$$

$$\int_{x^2}^{\infty} 2x e^{x^2-y}\, dy = 2xe^{x^2} \int_{x^2}^{\infty} e^{-y}\, dy$$

$$= 2xe^{x^2} \left[-e^{-y} \right]_{y=x^2}^{y=\infty}$$

$$= 2xe^{x^2} \left[0 - \left(-e^{-x^2} \right) \right]$$

$$= 2x \cancel{e^{x^2} \cdot e^{-x^2}} = 2x$$

$$\boxed{f_X(x) = \begin{cases} 2x & x \in (0,1) \\ 0 & \text{o.w.} \end{cases}}$$

$$\int_0^1 2x\, dx = x^2 \Big]_0^1 = 1$$

② $$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

$$= \int_{0}^{1} f_{X,Y}(x,y)\,dx$$

$\left(\because \; f_{X,Y} = 0 \text{ IF } x \notin (0,1) \right)$

$f_{X,Y} = 0$ IF $y \leq x^2$

IN PARTICULAR IF $y \leq 0$, $f_Y(y) = 0$

WE CAN ASSUME THAT $y \geqslant 0$

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx$$

$y \leq x^2 \Rightarrow \sqrt{y} \leq x$, $f_{X,Y} \equiv 0$

$\Rightarrow$ $f_{X,Y}$ WILL ONLY CONTRIBUTE IF

$$\boxed{x < \sqrt{y}}$$

CASE I : $y \geq 1$

$$\sqrt{y} \geq 1 \geq x$$

$$\Rightarrow \sqrt{y} \geq x$$

$$f_Y(y) = \int_0^1 2x e^{x^2 - y} \, dx = e^{-y} \int_0^1 2x e^{x^2} \, dx$$

$u = x^2$

$$= e^{-y} \int_0^1 e^u \, du$$

$$\int_0^1 e^u \, du = e - 1$$

$$f_Y(y) = e^{-y}\left[e-1\right] = e^{1-y} - e^{-y}$$

(WHEN $y \geq 1$)

CASE II : $0 \leq y < 1$

$\Rightarrow \quad 1 > \sqrt{y} \geq x$

$$\int_0^1 = \int_0^{\sqrt{y}} + \boxed{\int_{\sqrt{y}}^1}$$

$f_{X,Y} = 0$

$$f_Y(y) = \int_0^1 f_{X,Y}(x,y)\,dx$$

$$= \int_0^{\sqrt{y}} 2x\,e^{x^2-y}\,dx \qquad u = x^2$$

$$= \int_0^{y} e^{u-y}\,du = e^{u-y}\Big]_{u=0}^{u=y}$$

$$= 1 - e^{-y}$$

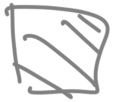
$\therefore$

$$f_Y(y) = \begin{cases} 0 & y \leq 0 \\ 1 - e^{-y} & 0 < y < 1 \\ e^{1-y} - e^{-y} & y \geq 1 \end{cases}$$

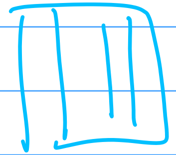
③ $P(Y < 3X^2) = \displaystyle\iint_{y < 3x^2} f_{X,Y}(x,y)\,dx\,dy$

$B = \{(x,y) : y < 3x^2\}$

L.H.S. $= P((X,Y) \in B) = \displaystyle\iint_{B} f_{X,Y}(x,y)\,dx\,dy$

$D =$ 

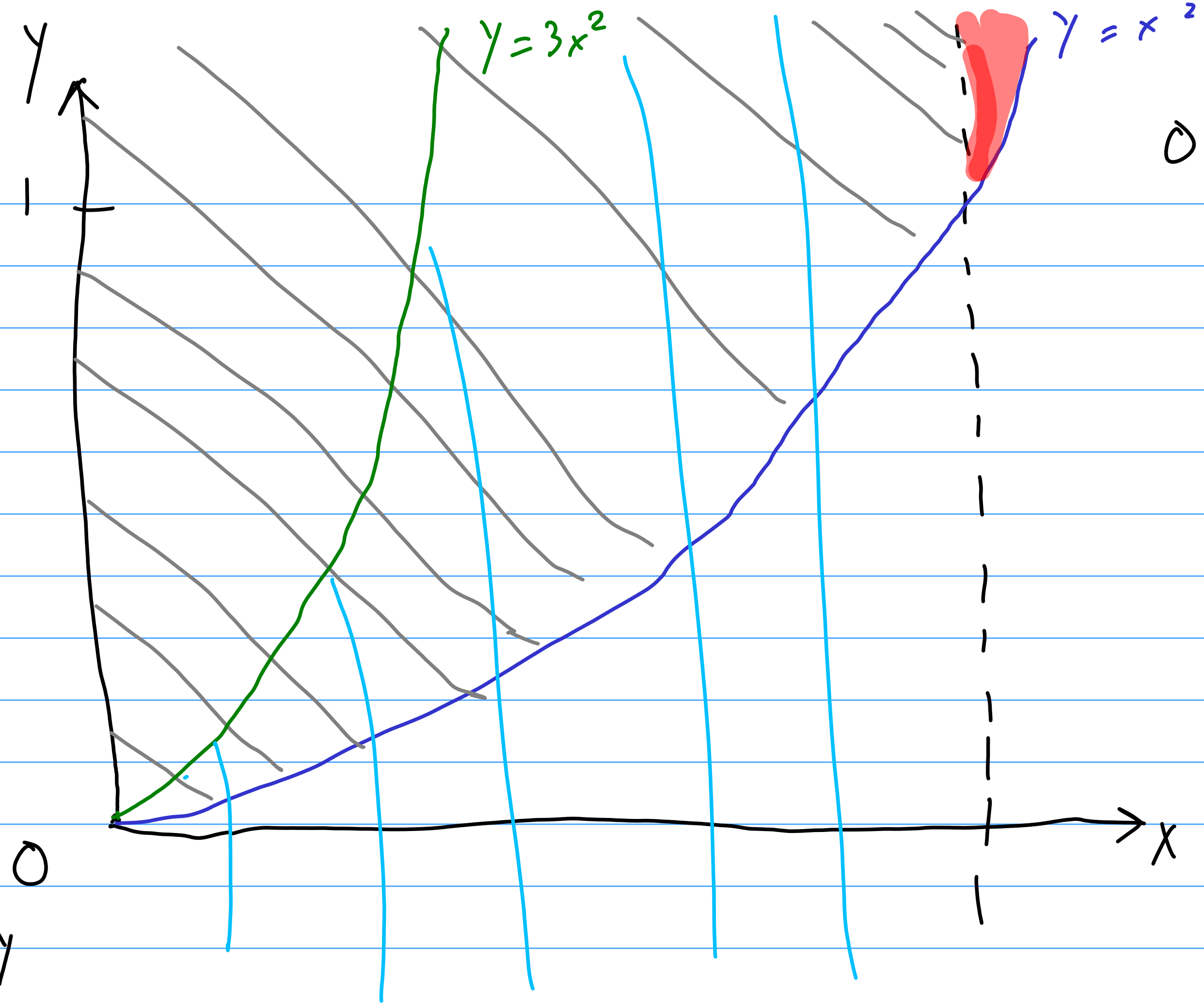
$= \text{WHERE } f_{x,y}(x,y)$

$B =$ 

$= \{ y < 3x^2 \}$

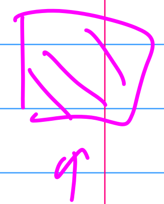
$$\iint_B f_{x,y}(x,y)\,dx\,dy$$

$$= \iint_{B \cap D} f_{x,y}(x,y)\,dx\,dy$$



$0 < x < 1$

$y > x^2$



$B \cap D$

$B \cap D$

$\downarrow$

TYPE I REGION

$x = 0$, $x = 1$

$y = 3x^2$, $y = x^2$

$$P(Y < 3X^2) = \iint_{B \cap D} f_{X,Y}(x,y)\, dx\, dy$$

(TYPE I)

$$= \int_{x=0}^{x=1} \left[\int_{y=x^2}^{y=3x^2} \left(2x e^{x^2 - y} \right) dy \right] dx$$

$$= \int_{x=0}^{x=1} \left(2x e^{x^2} \right) \left[\int_{y=x^2}^{y=3x^2} e^{-y}\, dy \right] dx$$

$$\int_{x^2}^{3x^2} e^{-y}\,dy = e^{-x^2} - e^{-3x^2}$$

[CHECK]

$$P(Y < 3X^2) = \int_0^1 2xe^{x^2}\left[e^{-x^2} - e^{-3x^2}\right] dx$$

$$= \int_0^1 \left[2x - 2xe^{-2x^2}\right] dx$$

$$1 \Leftarrow \boxed{\int_0^1 2x\,dx}$$

$$= \boxed{\int_0^1 2x\,dx} - \int_0^1 2x\,e^{-2x^2}$$

$$\int_0^1 2x e^{-2x^2}\, dx = \int_0^1 e^{-2u}\, du$$

$(u = x^2)$

$$= \left. \frac{-e^{-2u}}{2} \right]_0^1 = \frac{1}{2} - \frac{e^{-2}}{2}$$

$$P(Y < 3X^2) = 1 - \left[\frac{1}{2} - \frac{e^{-2}}{2} \right] = \frac{1 + e^{-2}}{2}$$