MATH 201 (SUMMER 2023, SESH A2)

LECTURE 13 : 06/07/23

ANURAG SAHAY
OFF HRS: BY APPT (VIA ZOOM)

email: anuragsahay@rochester.edu

Zoom ID:
979-4693-0650

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. REMINDER: MIDTERM REGRADES WILL CLOSE ON TODAY AT 11 PM ET

2. OFFICE HOURS: R: 11:15 AM – 12:15 PM

3. UPCOMING DEADLINES:
   a. HW 6 – } TODAY!
   b. $\boxed{\text{WW 6}}$ – } UPDATED ON WEBWORK.
   c. WW 7 – SAT
   d. HW 7 – SUN

4. PLEASE FILL OUT MID-SEM FEEDBACK. ANONYMOUS, OPTIONAL

5. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

CONT. OF § 6.2

UNIFORM DISTRIBUTION IN HIGHER DIMS.

(/VECTOR)

$(X, Y) \to$ RANDOM POINT (/VECTOR) IN $\mathbb{R}^2$

RECALL :

$X \sim \text{Unif}([a,b])$ IF $P(X \in [c,d])$ DEPENDS ONLY ON LENGTH OF $[c,d] = d-c$ $([c,d] \subseteq [a,b])$

$$P(X \in [c,d]) = \frac{d-c}{b-a}, \Rightarrow f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a,b] \\ 0 & \text{o.w.} \end{cases}$$

LENGTH OF $[a,b]$

IF $D \subseteq \mathbb{R}^2$, THEN $(X,Y) \sim \text{Unif}(D)$

IF $(X,Y) \in D$ ALWAYS, AND NO POINT IN D IS PREFERRED

j.p.d.f. OF (X,Y)?

MORE GENERALLY, $(X_1, \ldots, X_n) \to$ RANDOM VECTOR IN $\mathbb{R}^n$

$D \subseteq \mathbb{R}^n$ $\quad (X_1, \ldots, X_n) \sim \text{Unif}(D)$

NOTE, p.d.f OF $\mathrm{Unif}([a,b])$ IS CONSTANT.

$$f_{X,Y}(x,y) = \begin{cases} \boxed{c} & (x,y) \in D \\ 0 & (x,y) \notin D \end{cases}$$

SHOULD NOT DEPEND ON (x,y)

WE KNOW, $1 = \int_{\mathbb{R}^2} f_{X,Y}(x,y)\,dx\,dy = \int_D c\,dx\,dy = c\,\mathrm{AREA}(D)$

$$\Rightarrow c = \frac{1}{\mathrm{AREA}(D)}$$

$$f_{X,Y,Z}(x,y,z) = \begin{cases} c & (x,y,z) \in B \\ 0 & \text{o.w.} \end{cases}$$

$$1 = \int_{\mathbb{R}^3} f(x,y,z)\, dx\,dy\,dz = \int_B c \, dx\,dy\,dz = c \cdot \mathrm{VOL}(B)$$

$$\Rightarrow c = \frac{1}{\mathrm{VOL}(B)}$$

UNIFORM DISTRIBUTION IN $\mathbb{R}^n$ $(n = 2, 3)$

Definition 6.18. Let D be a subset of the Euclidean plane $\mathbb{R}^2$ with finite area. Then the random point (X, Y) is **uniformly distributed on** D if its joint density function is

$$f(x,y) = \begin{cases} \dfrac{1}{\text{area}(D)}, & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \notin D. \end{cases} \tag{6.15}$$

COMPARE TO

$$f(x) = \begin{cases} \dfrac{1}{b-a} = \dfrac{1}{\text{LENGTH}(D)} \\ 0 \end{cases}$$

$[x \in D \,/\, x \notin D]$

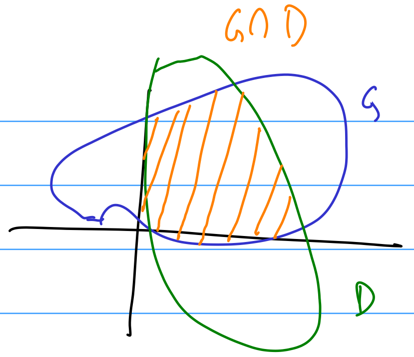
Let B be a subset of three-dimensional Euclidean space $\mathbb{R}^3$ with finite volume. Then the random point (X, Y, Z) is **uniformly distributed on** B if its joint density function is

$$f(x,y,z) = \begin{cases} \dfrac{1}{\text{vol}(B)}, & \text{if } (x,y,z) \in B \\ 0, & \text{if } (x,y,z) \notin B. \end{cases} \tag{6.16}$$

EXTENDS TO $\mathbb{R}^n$ $n \geq 4$

(REPLACE AREA/VOL. BY HYPERVOLUME)

FOR ANY $G \subseteq \mathbb{R}^2$



$$P((X,Y) \in G) = \int_G f(x,y)\,dx\,dy$$

[RECALL $f \equiv 0$ IF $(x,y) \notin D$] $$= \int_{G \cap D} \frac{1}{\text{AREA}(D)}\,dx\,dy = \frac{\text{AREA}(G \cap D)}{\text{AREA}(D)}$$

IN PARTICULAR, $G \subseteq D$

$$\Rightarrow P((X,Y) \in G) = \frac{\text{AREA}(G)}{\text{AREA}(D)}$$

COMPARE TO

$$P(X \in [c,d]) = \frac{d-c}{b-a}$$

$d-c$ → LEN $[c,d]$

$b-a$ → LEN $[a,b]$

$(X, Y, Z) \sim \text{Unif}(B)$

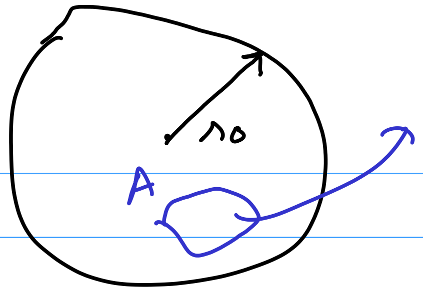
IIIly, FOR ANY $H \subseteq \mathbb{R}^3$

$$P((X,Y,Z) \in H) = \int_H f(x,y,z)\, dx\, dy\, dz$$

$$= \int_{H \cap B} \frac{1}{\text{VOL}(B)}\, dx\, dy\, dz \quad \left(\because f = 0 \text{ OUTSIDE } B\right)$$

$$= \frac{\text{VOL}(H \cap B)}{\text{VOL}(B)}$$

RECALL : DARTS



$\Pr(\text{DART LANDS IN } A) \propto \text{AREA}(A)$

Example 6.19. Let (X, Y) be a uniform random point on a disk D centered at $(0, 0)$ with radius r_0. (This example continues the theme of Example 3.19.) Compute the marginal densities of X and Y.

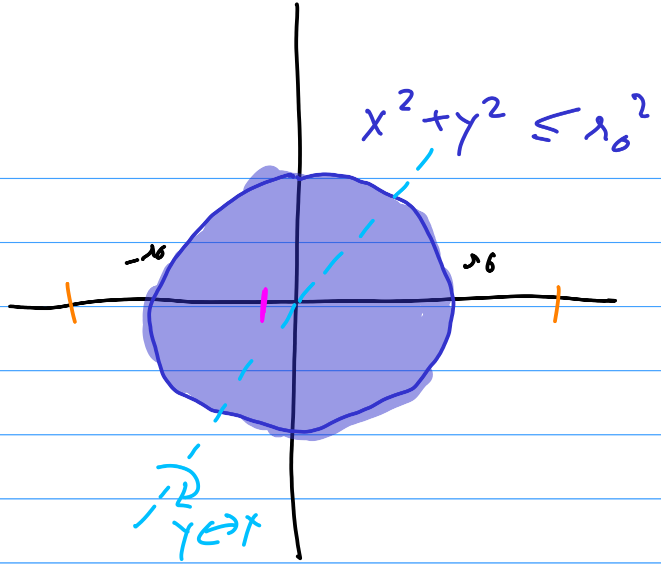
$(X,Y) \sim \text{Unif}(D)$, $D = \{(x,y) : x^2 + y^2 \le r_0^2\}$

$\text{AREA} = \pi r_0^2$

$$f_{X,Y}(x,y) = \begin{cases} 1/\pi r_0^2 & \text{IF } (x,y) \in D \rightsquigarrow x^2+y^2 \le r_0^2 \\ 0 & \text{o.w.} \end{cases}$$

$f_X , \quad f_Y$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy$$



$= 0$ IF $x < -r_0$ OR $x > -r_0$

LET'S ASSUME $x \in [-r_0, r_0]$

WHENEVER $f \neq 0$ $\Longrightarrow$ $x^2 + y^2 \le r_0^2$ $\Rightarrow$ $y^2 \le \boxed{r_0^2 - x^2}$

$\geq 0 \; (\because |x| \le r_0)$

$$\Rightarrow -\sqrt{r_0^2 - x^2} \le y \le \sqrt{r_0^2 - x^2}$$

$$f_X(x) = \int_{-\sqrt{r_0^2 - x^2}}^{\sqrt{r_0^2 - x^2}} \left(\frac{1}{\pi r_0^2} \right) dy$$

$$= \frac{1}{\pi r_0^2} \left[2\sqrt{r_0^2 - x^2} \right] = \frac{2}{\pi} \frac{\sqrt{r_0^2 - x^2}}{r_0^2} \quad \left(\text{IF } |x| \le r_0 \right.$$

$$\therefore \quad f_X(x) = \begin{cases} \dfrac{2}{\pi} \dfrac{\sqrt{r_0^2 - x^2}}{r_0^2} & \text{IF } -r_0 \le x \le r_0 \\ 0 & \text{o.w.} \end{cases}$$

(NOT EVEN CLEAR $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$ $\rightarrow$ CHECK!)

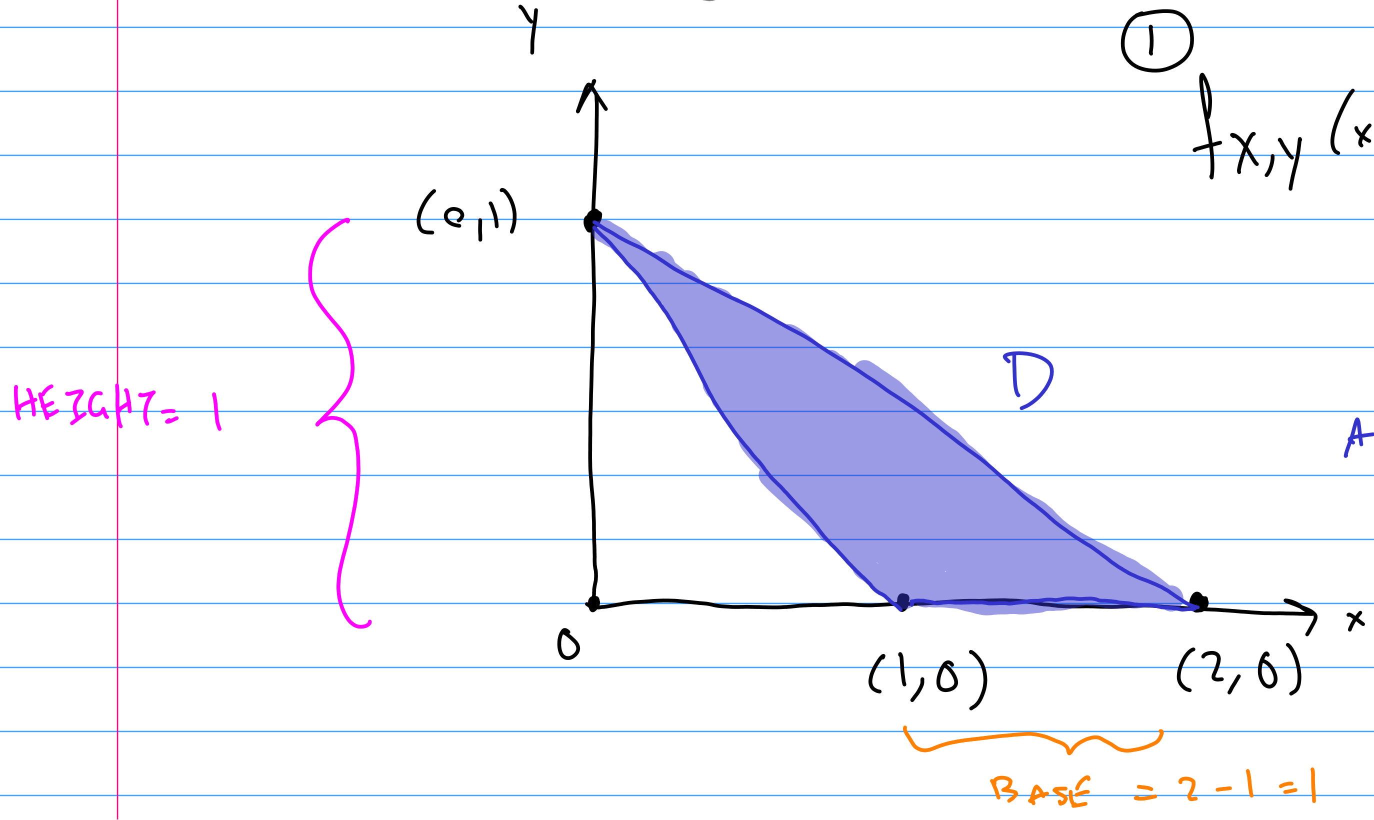
$$f_Y(y) = \begin{cases} \dfrac{2}{\pi} \cdot \dfrac{\sqrt{r_0^2 - y^2}}{r_0^2} & -r_0 \le y \le r_0 \\ 0 & \text{o.w.} \end{cases}$$

(BY SYMMETRY OF $X \leftrightarrow Y$)

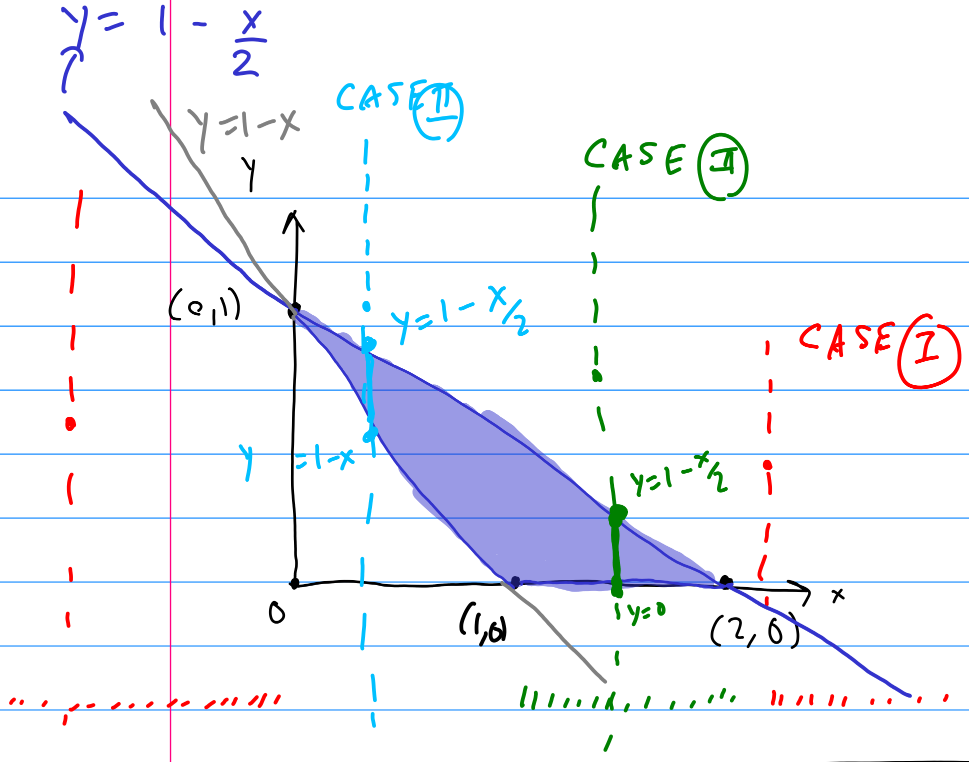
Example 6.20. Let (X, Y) be uniformly distributed on the triangle D with vertices $(1, 0)$, $(2, 0)$ and $(0, 1)$. (See Figure 6.3.) Find the the ① joint density function of (X, Y) and the marginal density functions of both ② X and ③ Y. Next, let A be the (random) area of the rectangle with corners $(0, 0)$, $(X, 0)$, $(0, Y)$, and (X, Y). Find ④ $E[A]$.

①

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\text{AREA}(D)} = 2 & (x,y) \in D \\ 0 & \text{o.w.} \end{cases}$$



$$\text{AREA}(D) = \frac{1}{2}(1)\cdot(1) = \frac{1}{2}$$



2

CASE I : $x > 2$ OR

$x < 0$

$$f_{X,Y}(x,y) = 0$$

$$\Rightarrow f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = 0$$

CASE II : $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^{1-x/2} 2\,dy = 2\left(1 - \frac{x}{2}\right)$

$\downarrow$

$1 \leq x \leq 2$

$$= 2 - x$$

CASE III : $0 \leq x \leq 1$

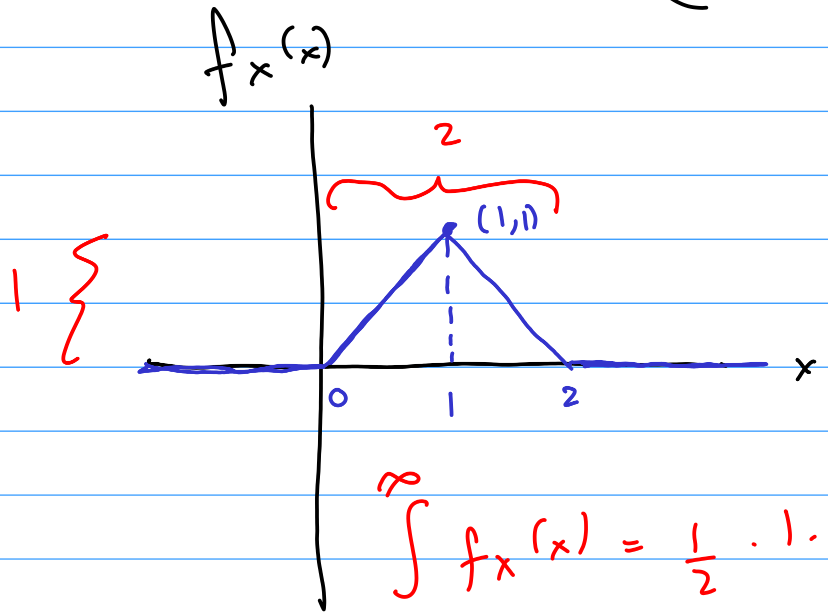
$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_{1-x}^{1-x/2} 2 \, dy = 2\left(1 - x/2\right) - 2(1-x)$$

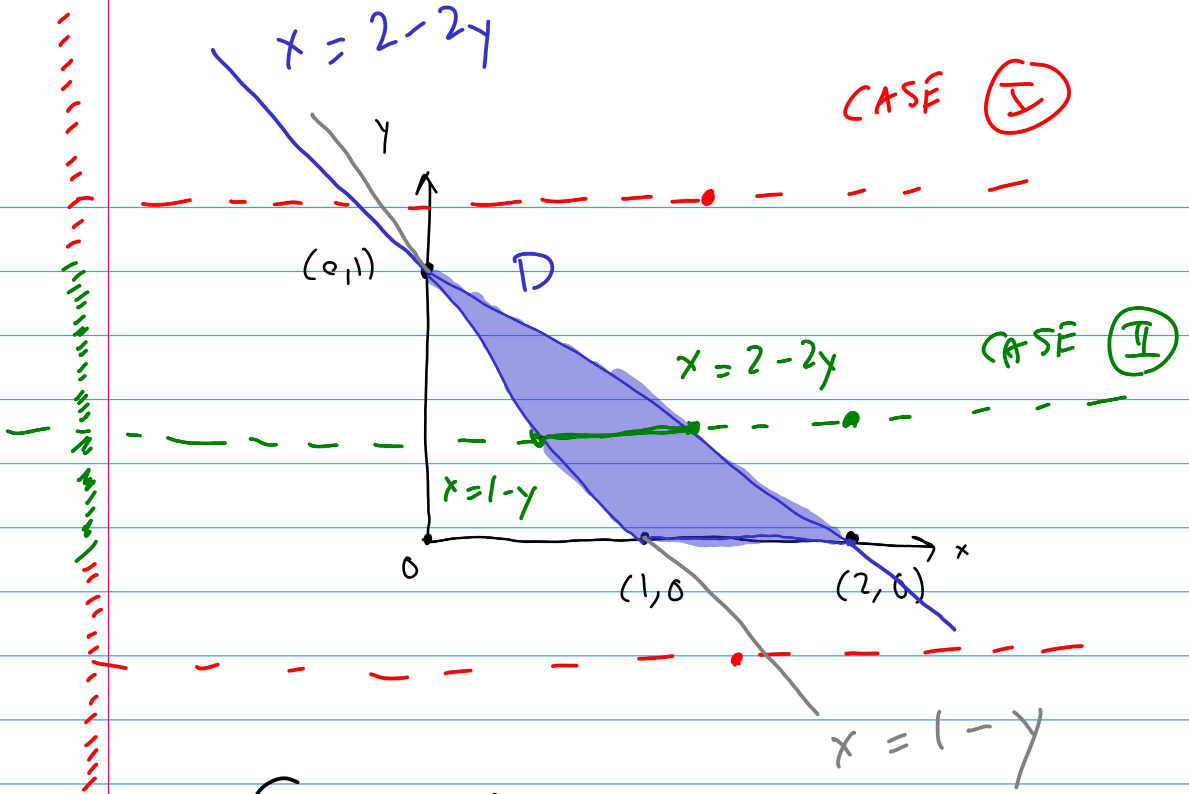
$$= x$$

$$f_X(x) = \begin{cases} 2-x, & 1 \le x \le 2 \\ x, & 0 \le x \le 1 \\ 0 & \text{o.w. (i.e. } x > 2 \text{ or } x < 0) \end{cases}$$

$$\boxed{\text{MARGINAL OF } X.}$$



$$\int_{-\infty}^{\infty} f_X(x) = \frac{1}{2} \cdot 1 \cdot 2 = 1$$



③ **CASE I** : $y > 1$ OR $y < 0$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = 0$$

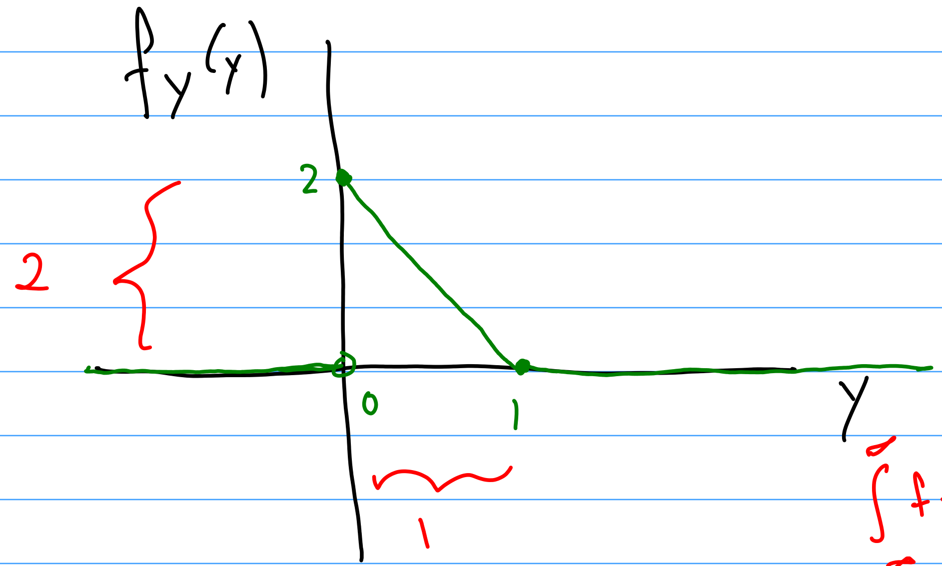
CASE II : $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_{1-y}^{2-2y} 2\,dx = 2(2-2y) - 2(1-y)$$

$$= 2 - 2y$$

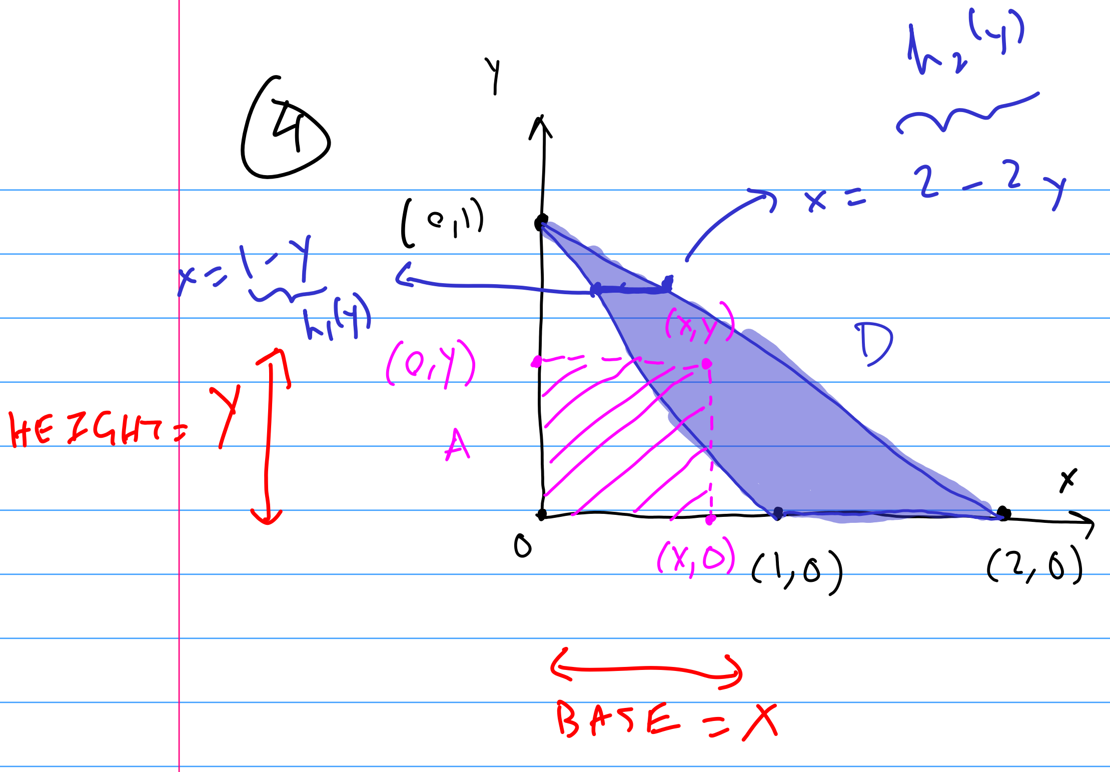
$$f_Y(y) = \begin{cases} 2-2y & 0 \le y \le 1 \\ 0 & \text{o.w. (i.e. } y > 1 \text{ OR } y < 0) \end{cases}$$

$\boxed{\text{MARGINAL OF } Y}$



$$\int_{-\infty}^{\infty} f_Y(y)\,dy = \frac{1}{2} \cdot 1 \cdot 2 = 1$$

(4)



A → AREA OF THE PINK RECTANGLE

$\mathbb{E}(A)$?

$A = XY$

$$\mathbb{E}(XY) = \iint_{\mathbb{R}^2} x \cdot y \cdot \underbrace{f_{X,Y}(x,y)}_{\text{SUPPORTED IN } D} \, dx\, dy = \iint_D (2xy)\, dx\, dy$$

DOUBLE INTEGRAL IS BOTH TYPE I & TYPE II.

HOWEVER, TYPE II CALC. IS EASIER

$$\iint_D (2xy)\,dx\,dy = \int_{y=0}^{y=1} \left[\int_{x=1-y}^{x=2-2y} (2xy)\,dx \right] dy$$

$$= \int_{y=0}^{y=1} \left[x^2 y \right]_{x=1-y}^{x=2-2y} dy = \int_0^1 \left[(2-2y)^2 y - (1-y)^2 y \right] dy$$

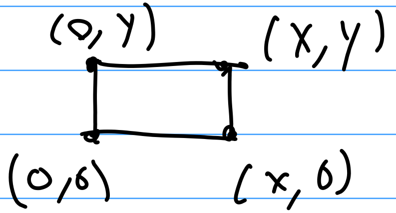
$$\int_0^1 \left[(2-2y)^2 y - (1-y)^2 y\right] dy = \int_0^1 \left[4 \quad (1-y)^2 y - (1-y)^2 y\right] dy$$

$$= 3\int_0^1 (1-y)^2 y \, dy$$

$$= \int_0^1 3 \left[y - 2y^2 + y^3\right] dy$$

$$= \left[\frac{3y^2}{2} - 2y^3 + \frac{3y^4}{4}\right]_{y=0}^{y=1} = \frac{3}{2} - 2 + \frac{3}{4} = \frac{1}{4}$$

EXPECTED AREA



$$= E(A) = \frac{1}{4}$$

BREAK TILL

10 : 20 AM

§ 6.3 JOINT DISTRIBUTIONS & INDEPENDENCE.

RECALL

$X_1, \ldots, X_n$ IS INDEPENDENT

IF FOR ALL REASONABLE $B_1, \ldots, B_n \subseteq \mathbb{R}$

JOINT DIST.

$$\underbrace{P(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n)} = P(X_1 \in B_1) \cdot P(X_2 \in B_2) \cdots P(X_n \in B_n)$$

RECALL, YOU DON'T NEED TO CHECK EVERY CONDITION.

Fact 6.22. Let $p(k_1, \ldots, k_n)$ be the joint probability mass function of the discrete random variables $X_1, \ldots, X_n$. Let $p_{X_j}(k) = P(X_j = k)$ be the marginal probability mass function of the random variable X_j. Then $X_1, \ldots, X_n$ are independent if and only if

$$p(k_1, \ldots, k_n) = p_{X_1}(k_1) \cdots p_{X_n}(k_n) \tag{6.21}$$

for all possible values $k_1, \ldots, k_n$.

[JOINT $= \prod$(MARGINALS)]

<u>NOTE</u> : WE ALREADY MADE THIS DEFN!

Example 6.24. Roll two fair dice. Let X_1 and X_2 be the outcomes. We checked in Example 2.30 that X_1 and X_2 are independent. Let $S = X_1 + X_2$. Determine whether X_1 and S are independent.

$X_1, X_2 \in \{1, 2, 3, 4, 5, 6\}$

$S = X_1 + X_2 \quad \in \{2, \ldots, 12\}$

INTUITIVELY: NO!

$P_S(2) = P(S = 2) = P(X_1 = 1, X_2 = 1) = \frac{1}{36}$

$P_{X_1}(2) = P(X_1 = 2) = \frac{1}{6}$

$$p_{S,X_1}(2,2) = P(S=2, X_1=2)$$

$$= P(X_1+X_2=2, X_1=2) = P(\underbrace{X_2=0}_{\text{NEVER HAPPENS}}, X_1=2)$$

$$= 0$$

$$p_{S,X_1}(2,2) = 0 \neq \frac{1}{36}\cdot\frac{1}{6} = p_S(2)\, p_{X_1}(2)$$

Fact 6.25. Let $X_1, \ldots, X_n$ be random variables on the same sample space. Assume that for each $j = 1, 2, \ldots, n$, X_j has density function f_{X_j}.

(a) If $X_1, \ldots, X_n$ have joint density function

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n) \tag{6.22}$$

then $X_1, \ldots, X_n$ are independent. [JOINT $= \prod$ (MARGINAL)]

(b) Conversely, if $X_1, \ldots, X_n$ are independent, then they are jointly continuous with joint density function

$$f(x_1, x_2, \ldots, x_n) = f_{X_1}(x_1) f_{X_2}(x_2) \cdots f_{X_n}(x_n).$$

Pf WHEN $n = 2$:

$$P(X \in A, Y \in B) = \int_A \int_B f_{X,Y}(x, y)\, dy\, dx = \int_A \int_B f_X(x) f_Y(y)\, dy\, dx$$

$$= \int_A f_X(x) \underbrace{\left[\int_B f_Y(y)\, dy \right]} dx$$

$$= \underbrace{\left[\int_B f_Y(y)\, dy \right]}_{P(Y \in B)} \cdot \underbrace{\left[\int_A f_X(x)\, dx \right]}_{P(X \in A)}$$

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B) \Rightarrow \text{INDEPENDENCE.}$$

CONVERSELY,

$$P(X \in A, Y \in B) = P(X \in A)\, P(Y \in B)$$

$$= \left[\int_A f_X(x)\, dx \right] \cdot \underbrace{\left[\int_B f_Y(y)\, dy \right]}$$

$$= \int_A \left[\int_B f_Y(y)\, dy \right] \underbrace{f_X(x)}\, dx$$

$$= \int_A \int_B \boxed{f_X(x)\, f_Y(y)}\, dx\, dy$$

JOINT p.d.f. OF (X, Y)

$f_{X,Y} = f_X \cdot f_Y$

Example 6.26. Suppose that X, Y have joint density function

$$f(x,y) = \begin{cases} \frac{7}{\sqrt{2\pi}} e^{-x^2/2-7y}, & -\infty < x < \infty \text{ and } y > 0 \\ 0, & \text{else.} \end{cases}$$

Are X and Y independent? Find the probability $P(X > 2, Y < 1)$.

$$\left.\begin{aligned} f(x,y) &= \frac{7}{\sqrt{2\pi}} \boxed{e^{-x^2/2-7y}} \\ &= \left(\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right) \cdot \left(7e^{-7y}\right) \end{aligned}\right\} \text{ if } y > 0$$

$$\left. f(x,y) = 0 = \left(\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right) \cdot (0) \right\} \ y \leq 0$$

$$f_X(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \Rightarrow X \sim \mathcal{N}(0,1)$$

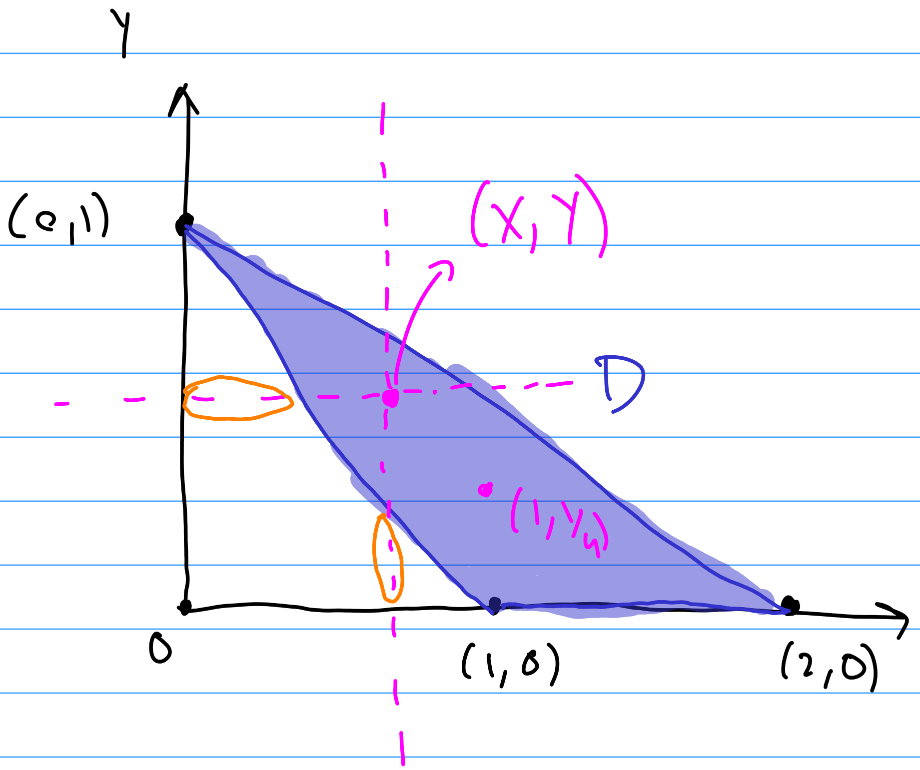
$$f_Y(y) = \left.\begin{cases} 7e^{-7y}, & y>0 \\ 0 & \text{o.w} \end{cases}\right\} \rightarrow Y \sim \text{Exp}(7)$$

$$f_{X,Y}(x,y) = f_X(x) \cdot f_Y(y)$$

$\therefore$ INDEPENDENT.

$$
\begin{aligned}
P(X > 2,\ Y < 1) &= P(X > 2) \cdot P(Y < 1) \\
&= P(2 < X < \infty) \cdot P(Y < 1) \\
&= \left[\Phi(\infty) - \Phi(2)\right] \cdot \left[1 - \underbrace{P(Y \geq 1)}_{e^{-7}}\right] \\
&= \left[1 - \Phi(2)\right] \cdot \left[1 - e^{-7}\right] \\
&= \left(1 - \Phi(2)\right) \cdot \left(1 - e^{-7}\right)
\end{aligned}
$$

Example 6.27. (Continuing Example 6.20) Recall the setup of Example 6.20 where we choose a random point uniformly from a triangle with vertices $(1, 0)$, $(2, 0)$ and $(0, 1)$. Check whether the random variables X and Y are independent or not.



EXPECT: NO!

INFO ABOUT $Y \to$ INFO ABOUT X

& VICE-VERSA

(e.g. $Y = 1/2 \Rightarrow X \neq 0.001$)

(e.g. $X = 1/2$, $Y \neq 1/4$)

RECALL : $f_{X,Y}(x,y) = \begin{cases} 2, & (x,y) \in D \\ 0, & \text{o.w} \end{cases}$

$$f_X(x) = \begin{cases} 2-x, & 1 \le x \le 2 \\ x, & 0 \le x \le 1 \\ 0, & \text{o.w.} \end{cases}$$

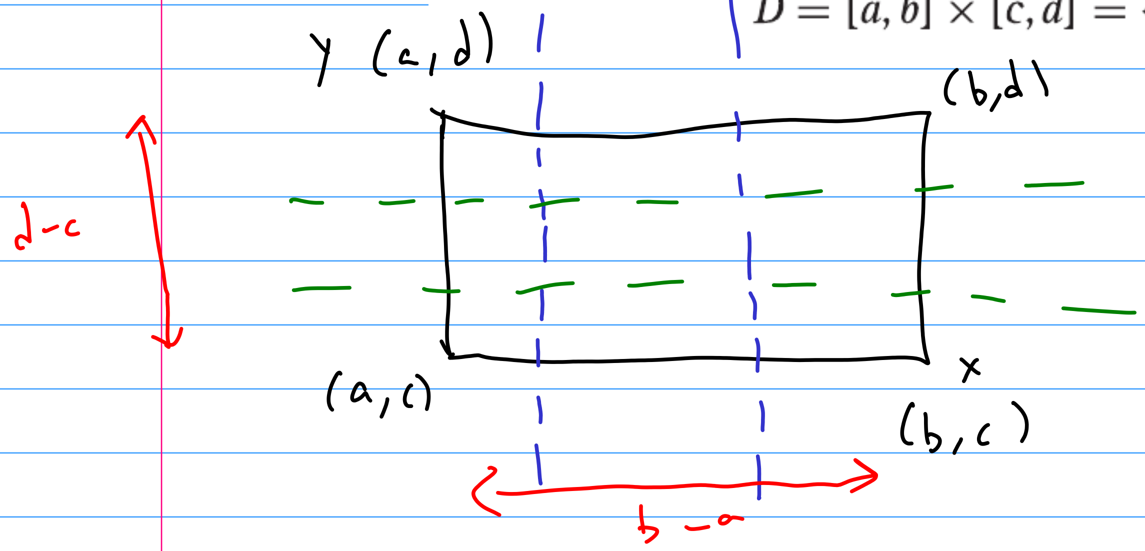
$$f_Y(y) = \begin{cases} 2-2y, & 0 \le y \le 1 \\ 0, & \text{o.w.} \end{cases}$$

$f_{X,Y}(1, 1/4) = 2$, $f_X(1) = 1$, $f_Y(1/4) = 2 - 2 \cdot 1/4 = 1.5$

$f_{X,Y}(1, 1/4) = 2 \neq 1 \times (1.5) = f_X(1)\, f_Y(1/4)$ #

Example 6.28. Let (X, Y) be a uniform random point on the rectangle

$$D = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}.$$



EXPECT INDEPENDENCE.

AREA $= (b-a)(d-c)$

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{(b-a)(d-c)}, & (x,y) \in D \longrightarrow x \in [a,b] \text{ \& } y \in [c,d] \\ 0 & \text{o.w.} \end{cases}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= 0 \qquad \left(\text{IF} \quad x \notin [a,b] \right)$$

IF $x \in [a,b)$,

$$f_X(x) = \int_c^d \frac{1}{(b-a)(d-c)}\,dy = \frac{1}{b-a}$$

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & o.w. \end{cases}$$

III l_y,

$$f_Y(y) = \begin{cases} \frac{1}{d-c}, & y \in [c,d] \\ 0, & o.w. \end{cases}$$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) = \begin{cases} \frac{1}{(b-a)(d-c)} & x \in [a,b] \ \& \ y \in [c,d] \\ 0 & o.w. \end{cases}$$

#

Fact 6.29. Suppose that $X_1, \ldots, X_{m+n}$ are independent random variables. Let $f : \mathbb{R}^m \to \mathbb{R}$ and let $g : \mathbb{R}^n \to \mathbb{R}$ be real-valued functions of multiple variables. Define random variables $Y = f(X_1, \ldots, X_m)$ and $Z = g(X_{m+1}, \ldots, X_{m+n})$. Then Y and Z are independent random variables.

Y, Z DO NOT HAVE INPUTS IN COMMON

Example 6.30. Consider a trial with success probability p that is repeated $m+n$ times. Let S be the number of successes among the first m trials, T the number of successes among the last n trials, and $Z = S + T$ the total number of successes. Check whether S and T are independent and check whether S and Z are independent.

$\text{Ber}(p) \rightarrow$ REPEATED $m+n$ TIMES.

$S =$ # SUCCESSES IN THE FIRST m TRIALS.

$T =$ # SUCCESSES IN THE LAST n TRIALS.

$Z = \underbrace{S + T}_{\text{TOTAL SUCC.}}$

$X_1, X_2, \ldots, X_{m+n}$ BE THE TRIALS.

$X_j \sim \text{Ber}(p)$, X_j ARE INDEPENDENT.

$$\Rightarrow \quad S = X_1 + X_2 + \cdots + X_m \qquad S \sim Bin(m, p)$$

$$T = X_{m+1} + X_{m+2} + \cdots + X_{m+n} \qquad T \sim Bin(n, p)$$

$$Z = X_1 + X_2 + \cdots + X_{m+n} \qquad Z \sim Bin(m+n, p)$$

$$\left.\begin{array}{l} S = f(X_1, \ldots, X_n) \\ T = f(X_{m+1}, \ldots, X_{m+n}) \end{array}\right\} \rightarrow \text{INDEPENDENT!}$$

Q. ARE S & Z INDEPENDENT?

ANSWER SHOULD BE NO.

$Z = S + T$!!!

$$P(Z = n-1) = p_Z(n-1) = \binom{m+n}{n-1} \cdot \underbrace{p^{n-1}(1-p)^{m+1}}_{\neq 0} \qquad (Z \sim Bin(m+n, p))$$

$$P(T = n) = p_T(n) = \binom{n}{n} \cdot \underbrace{p^n}_{\neq 0} \qquad (T \sim Bin(n, p))$$

$$P(S=n)\cdot P(Z=n-1) \neq 0$$

$$P_{T,Z}(n,n-1) = P(T=n,\ Z=n-1) = 0 \qquad (\because P(S=-1)=0)$$

$\in \{0, \ldots, m\}$

$$\underbrace{Z}_{n-1} = S + \underbrace{T}_{n} \Rightarrow S = (n-1) - n = -1$$

$$P_{T,Z} \neq P_T \cdot P_Z$$

$\Rightarrow$ NOT INDEPENDENT.