MATH 201 (SUMMER 2023, SESH A2)

LECTURE 14 : 06/08/23

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE
https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

(1) OFFICE HOURS : TODAY : 11:15 AM – 12:15 PM

(2) UPCOMING DEADLINES :
a. WW 7 – SAT
b. HW 7 – SUN
c. WW 8 – WED

(3) PLEASE FILL OUT MID-SEM FEEDBACK. → ANONYMOUS → OPTIONAL

(4) PLEASE KEEP VIDEOS ON, IF POSSIBLE !

RECALL:

(1) $X_1, X_2, \ldots, X_n$ ARE DISCRETE,

THEN $\{X_j\}$ IS INDEPENDENT

IF & ONLY IF

$$p_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = p_{X_1}(x_1)\cdots p_{X_n}(x_n)$$

(2) $X_1, \ldots, X_n$ ARE JOINTLY CONT., THEN
INDEPENDENT IF & ONLY IF

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_{X_1}(x_1)\cdots f_{X_n}(x_n)$$

Example 6.31. Suppose that $X_1, \ldots, X_n$ are independent random variables with $X_i \sim \mathrm{Geom}(p_i)$. Find the probability mass function of $Y = \min(X_1, \ldots, X_n)$, the minimum of $X_1, \ldots, X_n$.

$k \in \{1, 2, 3, \ldots\}$

$$P(X_j = k) = (1-p_j)^{k-1} p_j$$

$$Y = \min(X_1, \ldots, X_n)$$

1. $Y > k \iff X_1 > k, X_2 > k, \ldots, X_n > k$

$$P(Y>k) = P(X_1 > k, X_2 > k, \ldots, X_n > k)$$

$$= P(X_1 > k) \cdot P(X_2 > k) \cdots P(X_n > k)$$

$(\because X_j$ ARE IND.$)$

$$P(X_j > k) = (1 - p_j)^k$$

TAIL OF A GEOM. R.V.

$$P(Y > k) = (1-p)^k \iff Y \sim \text{Geom}(p)$$

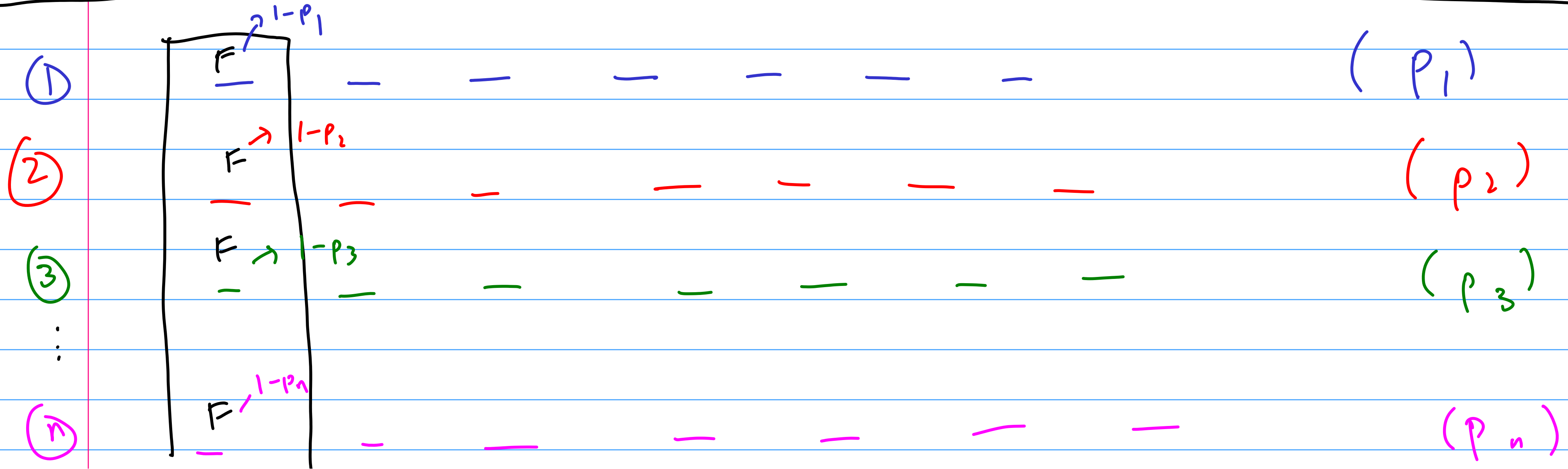
$$\underbrace{\underline{F}\ \ \underline{F}\ \ \underline{F}\ \ \underline{F}\ \ \underline{F}\ \ \underline{F}}_{\text{FIRST } k \text{ FLIPS ARE FAILURES.}}\ \ \underset{k+1}{\underline{?}}\ \ \underline{?}\ \ \underline{?}\ \ \underline{?}$$

$$P(Y > k) = \prod_{j=1}^{n} P(X_j > k) = \prod_{j=1}^{n} (1 - p_j)^k$$

$$= \Big[1 - \overbrace{\Big(1 - \prod_{j=1}^{n} (1 - p_j) \Big)}^{p} \Big]^k$$

$$P(Y > k) = (1-p)^k \Rightarrow Y \sim \text{Geom}(p).$$

$$p = 1 - \prod_{j=1}^{n} (1 - p_j)$$



IN EACH STEP, PROB. OF NOT STOPPING

$$= \prod_{j=1}^{n} (1 - p_j)$$

$\therefore$ PROB. OF STOPPING $= 1 - \overbrace{\prod_{j=1}^{n} (1 - p_j)}^{P}$

$P \quad P$

$$Y \sim \text{Geom}\left(1 - \prod_{j=1}^{n} (1 - p_j)\right)$$

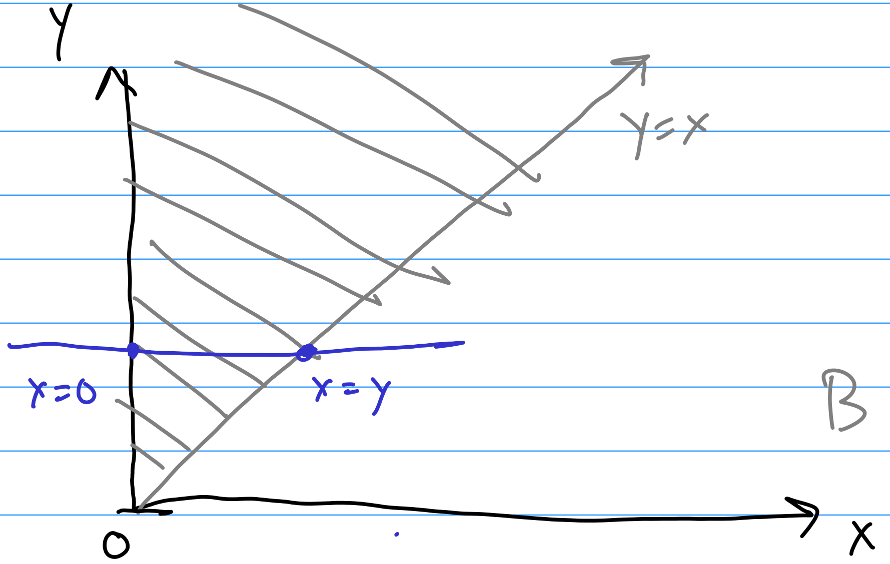
Example 6.33. You get phone calls from your mother and your grandmother independently. Let X be the time until the next call from your mother, and Y the time until the next call from your grandmother. Let us assume the distributions $X \sim \mathrm{Exp}(\lambda)$ and $Y \sim \mathrm{Exp}(\mu)$. Find the joint density $f_{X,Y}(x, y)$ of the pair (X, Y). What is the probability that your mother calls before your grandmother?

SIM. TO PREVIOUS BUT
① CONT.
② $n=2$

$X \sim \mathrm{Exp}(\lambda) \qquad Y \sim \mathrm{Exp}(\mu)$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$
$$= \left[\lambda e^{-\lambda x}\right] \cdot \left[\mu e^{-\mu y}\right] \qquad (\text{p.d.f. of } \mathrm{Exp}(\cdot))$$
$$= \begin{cases} \lambda \mu e^{-\lambda x - \mu y} & (x, y > 0) \\ 0 & (\text{IF } x < 0 \text{ OR } y < 0) \end{cases}$$

$$P(X < Y) = \iint_B f_{X,Y}(x,y)\,dx\,dy$$



(TYPE I & TYPE II)

$$B = \{x < y\}$$

$$B = \int_{y=0}^{y=\infty} \int_{x=0}^{x=y}$$

$$P(X<Y) = \int_0^\infty \int_0^y f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^\infty \int_0^y \lambda\mu\, e^{-\lambda x - \mu y}\,dx\,dy$$

$$= \int_0^\infty \mu e^{-\mu y} \left[\int_0^y \lambda e^{-\lambda x}\,dx \right] dy$$

$$= \left. -e^{-\lambda x} \right]_{x=0}^{x=y} = 1 - e^{-\lambda y}$$

$$P(X<Y) = \int_{y=0}^{y=\infty} \mu e^{-\mu y}\left[1 - e^{-\lambda y}\right] dy$$

$$= \int_0^\infty \mu \cdot e^{-\mu y}\, dy - \int_0^\infty \mu e^{-(\mu+\lambda) y}\, dy$$

$$\boxed{\int_0^\infty e^{-st}\, dt = \frac{1}{s}} \qquad = 1 - \frac{\mu}{\mu+\lambda} = \frac{\lambda}{\mu+\lambda}$$

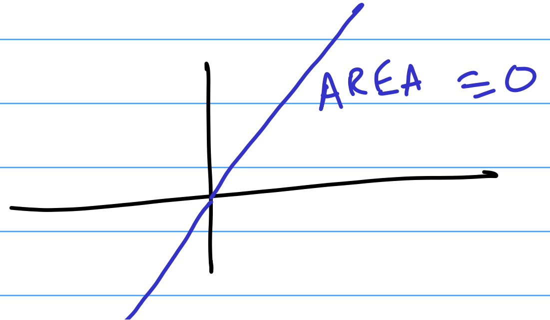
Ex.

$$P(X<Y) = \frac{\lambda}{\mu+\lambda}$$

$$P(Y>X) = 1 - P(X \leq Y)$$

$$= 1 - P(X<Y) = 1 - \frac{\lambda}{\mu+\lambda} = \frac{\mu}{\mu+\lambda}$$

($P(X=Y) = 0$ IF (X,Y) ARE JOINTLY CONT.)



$$(X, \lambda) \underset{\text{SYM}}{\longleftrightarrow} (Y, \mu)$$

$$P(X < Y) = \frac{\lambda}{\mu + \lambda} \Rightarrow P(Y < X) = \frac{\mu}{\lambda + \mu}$$

$\blacksquare$

Example 6.34. Continuing the previous example, let $T = \min(X, Y)$. T is the time until the next phone call comes from either your mother or your grandmother. Let us find the probability distribution of T.

EXPECT $T \sim \text{Exp}$!

WHY : (Geom, DISCRETE) $\longrightarrow$ (Exp, CONT.)

IDEA : LOOK AT TAILS.

$$P(T > t) = P(X > t, Y > t) \qquad \left(\because T = \min(X,Y) \right)$$

$$\{T > t\} = \{X > t, Y > t\}$$

$$= P(X > t)\, P(Y > t)$$

$\rightarrow$ (X,Y) ARE IND.

RECALL :

$$\boxed{\text{TAIL OF } S \sim \text{Exp}(s) \quad P(S>t) = e^{-st}} \text{ — } A$$

$$P(T>t) = \underbrace{P(X>t)}_{\text{Exp}(\lambda)}\ \underbrace{P(Y>t)}_{\text{Exp}(\mu)} = \left(e^{-\lambda t}\right)\cdot\left(e^{-\mu t}\right)$$

$$= e^{-(\lambda+\mu)t}$$

$$\boxed{P(T>t) = e^{-(\lambda+\mu)t}} \text{ — } B$$

COMPARING A TO B, T HAS THE SAME TAIL AS $\text{Exp}(\lambda+\mu)$

$\Rightarrow \quad T \sim Exp(\lambda + \mu)$

BREAK TILL
9:45 AM

§ 8.1 LINEARITY OF EXPECTATION

Fact 8.1. (Linearity of expectation) Let $g_1, g_2, \dots, g_n$ be single variable functions and $X_1, X_2, \dots, X_n$ random variables defined on the same sample space. Then

$$E[g_1(X_1)+g_2(X_2)+\cdots+g_n(X_n)] = E[g_1(X_1)]+E[g_2(X_2)]+\cdots+E[g_n(X_n)], \quad (8.3)$$

provided that all the expectations are finite. In particular,

$$E[X_1+X_2+\cdots+X_n] = E[X_1]+E[X_2]+\cdots+E[X_n], \quad (8.4)$$

provided all the expectations are finite.

($\mathbb{E}$ & Σ CAN BE INTERCHANGED)

NOTE: NO ASSUMPTION OF INDEPENDENCE!

$\underline{\text{Pf}}$: $n=2$, DISCRETE

$$\mathbb{E}\Big[\underbrace{g_1(X_1)+g_2(X_2)}_{\text{FUNC}(X_1,X_2)}\Big] = \sum_{k,l}\big[\underline{g_1}(k)+g_2(l)\big]\,p_{\underline{X_1,X_2}}(k,l)$$

$k \to$ VALUES X_1 TAKES

$l \to$ VALUES X_2 TAKES

$$= \sum_{k,l} g_1(k)\,p_{X_1,X_2}(k,l) + \sum_{k,l} g_2(l)\,p_{X_1,X_2}(k,l)$$

$$= \Big[\sum_k g_1(k) \overbrace{\Big[\sum_l p_{X_1,X_2}(k,l)\Big]}^{p_{X_1}(k)}\Big] + \Big[\sum_l g_2(l) \overbrace{\Big[\sum_k p_{X_1,X_2}(k,l)\Big]}^{p_{X_2}(l)}\Big]$$

$$\sum_{l} p_{X_1, X_2}(k, l) = p_{X_1}(k)$$

[MARGINAL $\Rightarrow$ SUMMING OVER OTHER R.V.S]

$$\mathbb{E}[g_1(X_1) + g_2(X_2)] = \underbrace{\left[\sum_{k} g_1(k)\, p_{X_1}(k)\right]}_{\mathbb{E}(g_1(X_1))} + \underbrace{\left[\sum_{l} g_2(l)\, p_{X_2}(l)\right]}_{\mathbb{E}(g_2(X_2))}$$

$$\therefore \mathbb{E}[g_1(X_1) + g_2(X_2)] = \mathbb{E}(g_1(X_1)) + \mathbb{E}(g_2(X_2))$$ ∎

GENERAL: $n > 2 \rightarrow$ INDUCTION.

X_j ARE CONT. $\rightarrow$ Σ, p.m.f. ARE REPLACED BY $\int$, p.d.f.

Example 8.2. Adam must pass both a written test and a road test to get his driver's license. Each time he takes the written test he passes with probability $\boxed{\frac{4}{10}}$, independently of other tests. Each time he takes the road test he passes with probability $\boxed{\frac{7}{10}}$, also independently of other tests. What is the total expected number of tests Adam must take before earning his license?

R → # OF ROAD TESTS. $\qquad R \sim \text{Geom}(4/10)$

W → # OF WRITTEN TESTS. $\qquad T \sim \text{Geom}(7/10)$

$T = R + W$ → TOTAL OF # OF TESTS.

$\mathbb{E}(T) \longrightarrow$ COMPUTE $\boxed{p_T(k)}$ $\qquad \mathbb{E}(T) = \sum_k k\, p_T(k)$

↳ TRY THIS.

RECALL : $X \sim Geom(p) \Rightarrow \mathbb{E}(X) = \frac{1}{p}$

WE KNOW,

$$\mathbb{E}(R) = \frac{1}{4/10} = \frac{10}{4} = \frac{5}{2}$$

$$\mathbb{E}(W) = \frac{1}{7/10} = \frac{10}{7}$$

$$\mathbb{E}(T) = \mathbb{E}(R+W) = \mathbb{E}(R) + \mathbb{E}(W)$$

$$= \frac{5}{2} + \frac{10}{7} = \frac{55}{14} \approx 3.93$$

Example 8.3. (Expected value of the binomial, easy calculation) Let $X \sim \mathrm{Bin}(n, p)$. The linearity of the expectation yields a short proof that $E[X] = np$.

$$X \sim \mathrm{Bin}(n,p) \Rightarrow \mathbb{E}[X] = np$$

$$P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mathbb{E}(X) = \sum_k k \cdot P(X=k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boxed{j = k-1 \quad , \quad k \cdot \binom{n}{k} = n \binom{n-1}{k-1}}$$

$$X \sim \mathrm{Bin}(n, p)$$

$$X = X_1 + X_2 + \cdots + X_n \qquad X_j \sim \mathrm{Ber}(p),$$

INDEPENDENCE

$$X_j \in \{0, 1\}$$

$$\mathbb{E}(X_j) = \sum_k k \cdot P(X_j = k)$$

$$= \cancel{0 \cdot P(X_j = 0)} + 1 \cdot \underbrace{P(X_j = 1)}_{p} \Rightarrow \mathbb{E}(X_j) = p.$$

$$\therefore \mathbb{E}[X] = \mathbb{E}[X_1 + X_2 + \cdots + X_n] = \overbrace{\mathbb{E}(X_1)}^{p} + \overbrace{\mathbb{E}(X_2)}^{p} + \cdots + \overbrace{\mathbb{E}(X_n)}^{p} = np$$

PREV. SLIDE IS AN EXAMPLE OF THE

INDICATOR METHOD

IDEA: IF X IS A COUNTING R.V. (i.e. $X \in \{0, 1, 2, \dots\}$)
TRY TO WRITE

$$X = \sum_j I_{E_j}$$

WHERE

$E_j \rightarrow$ EVENT

$I_{E_j} \rightarrow$ INDICATOR FUNTION $\longrightarrow$ i.e. $I_{E_j}(\omega) = \begin{cases} 1 & \omega \in E_j \\ 0 & \omega \notin E_j \end{cases}$

RECALL : $I_E \sim \mathrm{Ber}(P(E))$

$$\left\{\begin{aligned} P(I_E = 1) &= P(\omega \in E) = P(E) \\ P(I_E = 0) &= P(\omega \notin E) = P(E^c) = 1 - P(E) \end{aligned}\right.$$

BERNOULLI.

$$\mathbb{E}[X] = \mathbb{E}\left[\sum_j I_{E_j}\right] = \sum_j \underbrace{\mathbb{E}[I_{E_j}]}_{\text{PROB. OF SUCCESS.}} = \sum_j P(E_j)$$

Example 8.5. There are 15 workers in the office. Management allows a birthday party on the last day of a month if there were birthdays in the office during that month. How many parties are there on average during a year? Assume for simplicity that the birth months of the individuals are independent and each month is equally likely.

P → # OF PARTIES A YEAR

$P \in \{1, \dots, 12\}$ [COUNTING]

E_j = A PARTY OCCURS IN MONTH j.

e.g. E_1 → PARTY OCCURS IN JAN, E_{12} → DEC.

$$I_{E_j} = \begin{cases} 1 & \text{IF PARTY IN MONTH } j \\ 0 & \text{o.w.} \end{cases}$$

$P = I_{E_1} + I_{E_2} + \dots + I_{E_{12}}$

L.O.E

$$\mathbb{E}[P] = \sum_{j=1}^{12} \mathbb{E}[I_{E_j}] = \sum_{j=1}^{12} P(E_j)$$

$$P(E_j) = 1 - \underbrace{P(E_j^c)}_{\downarrow} = 1 - \left(\frac{11}{12}\right)^{15}$$

P(AT LEAST ONE PERSON HAS A BIRTHDAY IN MONTH j)

P(NO ONE HAS BIRTHDAY IN MONTH j)

$$P(E_j^c) = P[A_1 \neq j, \ldots, A_{15} \neq j]$$

$$= \prod_{k=1}^{15} P(A_k \neq j) = \left(\frac{11}{12}\right)^{15}$$

A_k = BIRTH MONTH OF PERSON k $(k = 1, \ldots, 15)$

$$\mathbb{E}(P) = \sum_{j=1}^{12} P(E_j) = 12\left[1 - \left(\frac{11}{12}\right)^{15}\right]$$

$$\approx 8.75$$

4 SUITS, 13 CARDS / SUIT.

Example 8.6. We deal five cards from a deck of 52 ~~without replacement~~. Let X denote the number of aces among the chosen cards. Find the expected value of X.

TWO WAYS : 1 IS THE jth CARD AN ACE?

2 IS THE ith ACE INCLUDED?

$A \rightarrow$ # OF ACES

$$C_j = \begin{cases} 1 & \text{IF CARD } j \text{ IS AN ACE} \\ 0 & \text{O.W.} \end{cases}$$

$$A = C_1 + C_2 + C_3 + C_4 + C_5 \qquad C_1 \leftrightarrow C_2 \Leftrightarrow C_3 \Leftrightarrow C_4 \Leftrightarrow C_5$$

$$\mathbb{E}(C_1) = \mathbb{E}(C_2) = \dots = \mathbb{E}(C_5)$$

$$\mathbb{E}(C_1) = P(\text{FIRST CARD IS ACE})$$

$$= \frac{4}{52} = \frac{1}{13}$$

$$\mathbb{E}(C_3) = P(\text{THIRD CARD IS ACE})$$

$$= \frac{4 \cdot 51 \cdot 50}{52 \cdot 51 \cdot 50} = \frac{1}{13}$$

$$\mathbb{E}(A) = \sum_{j=1}^{5} \mathbb{E}(C_j) = \frac{5}{13}$$

SPADES $\rightarrow$ 1

HEARTS $\rightarrow$ 2

DIAMOND $\rightarrow$ 3

CLUBS $\rightarrow$ 4

$$A_j = \begin{cases} 1 & \text{IF ACE OF SUIT } j \text{ IS DRAWN} \\ 0 & \text{o.w.} \end{cases}$$

$$D = A_1 + A_2 + A_3 + A_4$$

$$E(D) = E(A_1) + \underbrace{E(A_2)}_{E(A_1)} + \underbrace{E(A_3)}_{E(A_1)} + \underbrace{E(A_4)}_{E(A_1)} = 4E(A_1)$$

$$\mathbb{E}(A_1) = P(\text{ACE OF SUIT } j \text{ IS DRAWN})$$

$$= \frac{\binom{51}{4}}{\binom{52}{5}} = \frac{(\cancel{51}\cdot\cancel{50}\cdot\cancel{49}\cdot\cancel{48})/\cancel{(1\cdot 2\cdot 3\cdot 4)}}{(52\cdot\cancel{51}\cdot\cancel{50}\cdot\cancel{49}\cdot\cancel{48})/(\cancel{1\cdot 2\cdot 3\cdot 4}\cdot 5)}$$

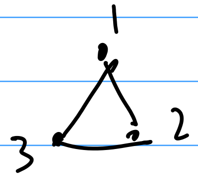
$$= \frac{5}{52}$$

$$\mathbb{E}(D) = 4\mathbb{E}(A_1) = \cancel{4}\cdot\frac{5}{\cancel{52}_{13}} = \frac{5}{13}$$

SKIP

Example 8.8. The sequence of coin flips $\underline{\mathrm{HH}}\mathrm{T}\underline{\mathrm{HHH}}\mathrm{TTT}\underline{\mathrm{H}}\mathrm{T}\underline{\mathrm{HHH}}\mathrm{T}$ contains two runs of heads of length three, one run of length two and one run of length one. The head runs are underlined. If a fair coin is flipped 100 times, what is the expected number of runs of heads of length 3?

Example 8.9. There are n guests at a party. Assume that each pair of guests know each other with probability $\frac{1}{2}$, independently of the other guests. Let X denote the number of groups of size three where everybody knows everybody else. Find $E[X]$.



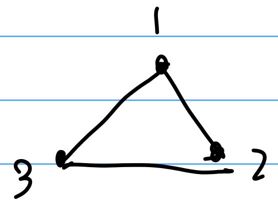
$$I_{a,b,c} = \begin{cases} 1 & \text{IF } a,b,c \text{ ALL KNOW EACH OTHER} \\ 0 & \text{O.W.} \end{cases}$$

$$X = \sum_{\{a,b,c\}} I_{a,b,c} \Rightarrow \mathbb{E}(X) = \sum_{\{a,b,c\}} \overbrace{\mathbb{E}(I_{a,b,c})}^{\mathbb{E}(I_{1,2,3})}$$

$$\mathbb{E}(X) = \sum_{\{a,b,c\}} \mathbb{E}(I_{1,2,3}) = \mathbb{E}(I_{1,2,3}) \cdot \binom{n}{3}$$

OF SUMMANDS = 3-ELEMENT SUBSETS.

$$\mathbb{E}(I_{1,2,3}) = P(1 \text{ KNOWS } 2,\ 2 \text{ KNOWS } 3,\ 3 \text{ KNOWS } 1)$$



$$= P(1 \text{ KNOW } 2)\, P(2 \text{ KNOWS } 3)\, P(3 \text{ KNOWS } 1)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$$

$$\boxed{\mathbb{E}(X) = \frac{1}{8}\binom{n}{3}}$$

BREAK TILL

10:50 AM

§ 8.2 EXPECTATION & INDEPENDENCE

Fact 8.10. Let $X_1, \ldots, X_n$ be independent random variables. Then for all functions $g_1, \ldots, g_n$ for which the expectations below are well defined,

$$E\left[\prod_{k=1}^{n} g_k(X_k)\right] = \prod_{k=1}^{n} E\left[g_k(X_k)\right]. \tag{8.6}$$

IN PARTICULAR, $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ IF X, Y IND.

$\boxed{\text{NOTE : INDEPENDENCE IS ASSUMED !}}$

PF FOR $n=2$, CONT.:

$X, Y \to$ IND. & JOINTLY CONT.

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

$$\mathbb{E}[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\,dx\,dy = \int_{-\infty}^{\infty} y\, f_Y(y) \underbrace{\boxed{\int_{-\infty}^{\infty} x\, f_X(x)\,dx}}_{\mathbb{E}(X)}\,dy$$

$$\mathbb{E}[XY] = \mathbb{E}(X) \boxed{\int_{-\infty}^{\infty} y f_Y(y)\, dy} \rightarrow \mathbb{E}(Y)$$

$$\mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y]$$

GENERAL : (1) $n > 2 \rightarrow$ INDUCTION

(2) $\int$, p.d.f $\rightarrow$ Σ, p.m.f.

IMPORTANT CONSEQUENCE

IMP.

Fact 8.11. (Variance of a sum of independent random variables) Assume the random variables $X_1, \dots, X_n$ are $\boxed{\text{independent}}$ and have finite variances. Then

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n). \quad (8.7)$$

<u>Pf for $n=2$</u> :

$\mathrm{Var}(X_1 + X_2)$

$\mu_1 = \mathbb{E}(X_1)$, $\mu_2 = \mathbb{E}(X_2)$

$\mathbb{E}(X_1 + X_2) = \mu_1 + \mu_2$

$$\mathrm{Var}(X_1 + X_2) = \mathbb{E}\left[\left\{(X_1 + X_2) - (\mu_1 + \mu_2)\right\}^2\right]$$

$$= \mathbb{E}\left[\left\{\overbrace{(X_1 - \mu_1)}^{a} + \overbrace{(X_2 - \mu_2)}^{b}\right\}^2\right]$$

$$\mathrm{Var}(X_1 + X_2) = \mathbb{E}\left[(X_1 - \mu_1)^2 + (X_2 - \mu_2)^2 + 2(X_1 - \mu_1)(X_2 - \mu_2) \right]$$

$$\overset{\text{L.O.E.}}{=} \underbrace{\mathbb{E}\left[(X_1 - \mu_1)^2\right]}_{\mathrm{Var}(X_1)} + \underbrace{\mathbb{E}\left[(X_2 - \mu_2)^2\right]}_{\mathrm{Var}(X_2)} + 2\underbrace{\mathbb{E}\left[(X_1 - \mu_1)(X_2 - \mu_2)\right]}_{0}$$

$$\mathbb{E}\left(\underbrace{(X_1 - \mu_1)}_{g_1(X_1)} \underbrace{(X_2 - \mu_2)}_{g_2(X_2)} \right) = \mathbb{E}(X_1 - \mu_1) \cdot \mathbb{E}(X_2 - \mu_2)$$

$$= \left[\underbrace{\mathbb{E}(X_1)}_{\mu_1} - \mu_1\right] \cdot \left[\underbrace{\mathbb{E}(X_2)}_{\mu_2} - \mu_2\right]$$

$$= 0.$$

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2)$$

$\blacksquare$

Example 8.12. (Variance of the binomial, easy calculation)

$X \sim \mathrm{Bin}(n,p)$ $\qquad \mathbb{E}(X) = np$

$\mathrm{Var}(X) = np(1-p)$

$$\mathbb{E}(X^2) = \sum_k k^2 \, P(X=k)$$

$$= \sum_{k=0}^{n} k^2 \binom{n}{k} p^k (1-p)^{n-k}$$

[RECALL : VERY HARD TO FOLLOW]

$$X = X_1 + X_2 + \cdots + X_n$$

$X_j \sim \mathrm{Ber}(p)$

$X_j \rightarrow$ INDEPENDENT.

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n)$$

$$\mathrm{Var}(X_j) = \mathbb{E}\left((X_j - \mu_j)^2\right) = \mathbb{E}\left((X_j - p)^2\right)$$

$$= \mathbb{E}\left[X_j^2 - 2pX_j + p^2\right]$$

$$X_j^2 = \begin{cases} 1 & X_j = 1 \\ 0 & X_j = 0 \end{cases}$$

$$= \mathbb{E}\left[X_j - 2pX_j + p^2\right]$$

$$= \mathbb{E}(X_j) - 2p\,\mathbb{E}(X_j) + p^2$$

$$E(X_j) = p$$

$$\mathrm{Var}(X_j) = p - 2p \cdot p + p^2 = p - p^2 = p(1-p)$$

$$\mathrm{Var}(X) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n)$$

$$= p(1-p) + \cdots + p(1-p) = np(1-p)$$