MATH 201 (SUMMER 2023, SESH A2)

LECTURE 15 : 06/12/23

ANURAG SAHAY
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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

(1) OFFICE HOURS : MW : 11:15 AM – 12:15 PM (BY APPT.)

→ EXTENSION.

(2) UPCOMING DEADLINES : (i) HW 7 – $\boxed{\text{TUES}}$, (iii) HW 8 – THURS

(ii) WW 8 – WED (iv) WW 9 – SAT

NEXT WEEK'S DEADLINES ARE ALSO UP!

(3) GRADING POLICY : EXTRA CREDIT – HW 10

CAN INCREASE UP TO 5% OF MAX.

(4) CLASS ON T, JUNE 20th TO BE FLIPPED.

(5) PLEASE KEEP VIDEOS ON, IF POSSIBLE!

RE: GRADING POLICY

→ HW 10 WILL NOT BE INCL. IN HOMEWORK (20%)

→ UP TO 5% OF THE MAX SCORE CAN BE INCREASED BY HW 10.

→ HW IS 50 POINTS. 10 H $\longrightarrow$ H% MORE IN FINAL SCORE

NEW SCORE = min (OLD SCORE + H, 100)

§ 8.4 COVARIANCE & CORRELATION

Definition 8.23. Let X and Y be random variables defined on the same sample space with expectations μ_X and μ_Y. The **covariance** of X and Y is defined by

$$\mathrm{Cov}(X, Y) = E\left[(X - \mu_X)(Y - \mu_Y)\right] \tag{8.18}$$

if the expectation on the right is finite. $\mathbb{E}\left[(X-\mathbb{E}X)(Y-\mathbb{E}Y)\right]$

NOTE :

$$\mathrm{Cov}(X,X) = \mathbb{E}\left(\left(X-\mu_X\right)\cdot\left(X-\mu_X\right)\right) = \mathbb{E}\left[\left(X-\mu_X\right)^2\right]$$

$$= \mathrm{Var}(X)$$

Fact 8.24. (Alternative formula for the covariance)

$$\mathrm{Cov}(X,Y) = E[XY] - \mu_X\mu_Y. \quad = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}(Y) \tag{8.20}$$

COMPARE $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

PF:
$$\mathrm{Cov}(X,Y) = \mathbb{E}\left[(X-\mu_X)\cdot(Y-\mu_Y)\right]$$
$$= \mathbb{E}\left[X\cdot Y - \mu_X\cdot Y - X\mu_Y + \mu_X\mu_Y\right]$$
(LOE)
$$= \mathbb{E}[XY] \boxed{- \mu_X \underbrace{\mathbb{E}(Y)}_{\mu_Y} - \mu_Y \underbrace{\mathbb{E}(X)}_{\mu_X} + \mu_X\mu_Y} \Rightarrow -\mu_X\mu_Y$$

$$\Rightarrow \mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mu_X\mu_Y$$

	$(X - \mu_x)$	$(Y - \mu_y)$	$(X - \mu_x)(Y - \mu_y)$
1	$+ \quad (X > \mu_x)$	$+ \quad (Y > \mu_y)$	$+$
2	$+ \quad (X > \mu_x)$	$- \quad (Y < \mu_y)$	$-$
3	$- \quad (X < \mu_x)$	$+ \quad (Y > \mu_y)$	$-$
4	$- \quad (X < \mu_x)$	$- \quad (Y < \mu_y)$	$+$

X, Y FLUCT. ON OPP. SIDES.

X, Y FLUCTUATE ON THE SAME SIDE OF MEAN

(+ve CORRELATED)

$Cov(X,Y) > 0 \Rightarrow$ 1 & 4 $\Rightarrow$ "POSITIVE CORRELATION"

(-ve CORRELATED)

$Cov(X,Y) < 0 \Rightarrow$ 2 & 3 $\Rightarrow$ "NEGATIVE CORRELATION"

$Cov(X,Y) = 0 \Rightarrow$ "NO CORRELATON / UNCORRELATED"

Example 8.25. (Covariance of indicator random variables) Let A and B be two events and consider their indicator random variables I_A and I_B. Find $\mathrm{Cov}(I_A, I_B)$.

ALSO: WHAT DOES CORRELATION OF I_A & I_B MEAN?

$$I_E(\omega) = \begin{cases} 1 & \text{IF } \omega \in E \\ 0 & \text{o.w.} \end{cases}$$

$$\mathrm{Cov}(I_A, I_B)$$

$$(I_A I_B)(\omega) = \begin{cases} 1 & \text{IF } \omega \in A \text{ \& } \omega \in B \\ 0 & \text{o.w.} \end{cases} = I_{A \cap B}(\omega)$$

$$\boxed{\therefore \; I_A I_B = I_{A \cap B}}$$

$$\mathrm{Cov}(I_A, I_B) = \underbrace{\mathbb{E}[I_A I_B]}_{\mathbb{E}[I_{A\cap B}]} - \mathbb{E}[I_A]\,\mathbb{E}[I_B]$$

RECALL: $\mathbb{E}[I_A] = P(A)$

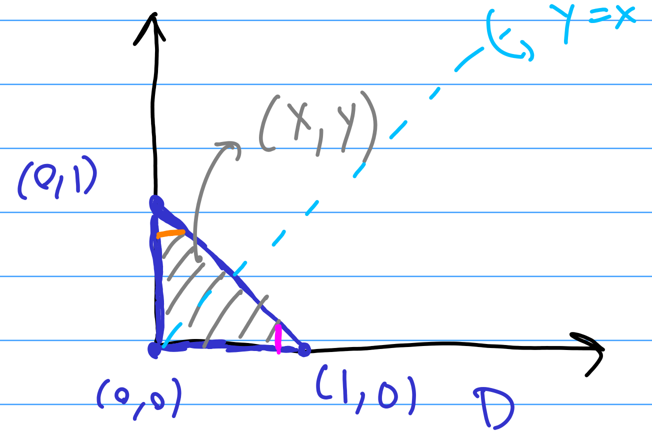
$$\therefore \quad \mathrm{Cov}(I_A, I_B) = P(A \cap B) - P(A)P(B)$$

$+ve$: $P(A \cap B) > P(A)P(B) \Rightarrow$ $P(A|B) > P(A)$, $P(B|A) > P(B)$

$-ve$: $P(A \cap B) < P(A)P(B) \Rightarrow$ $P(A|B) < P(A)$, $P(B|A) < P(B)$

$= 0$: $P(A \cap B) = P(A)P(B) \Rightarrow$ A & B ARE INDEP.

Example 8.26. Let (X, Y) be a random point uniformly distributed on the triangle D with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Calculate $\mathrm{Cov}(X, Y)$. Before calculation, the reader should determine from the geometry whether one should expect X and Y to be positively or negatively correlated.



NOTE: WHEN X IS BIG, Y IS FORCED TO BE SMALL.
(& VICE-VERSA)

$\therefore$ GUESS: $-$ve CORRELATION

$$\text{AREA OF } \triangle = \frac{1}{2}\cdot(1)\cdot(1) = \frac{1}{2} \Rightarrow f_{X,Y}(x,y) = \begin{cases} 2 & \text{IF } (x,y)\in D \\ 0 & \text{o.w.} \end{cases}$$

$$\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xy \, f_{X,Y}(x,y) \, dx\, dy$$

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x \, f_{X,Y}(x,y) \, dx\, dy$$

$$\mathbb{E}[Y] = \iint_{\mathbb{R}^2} y \, f_{X,Y}(x,y) \, dx\, dy$$

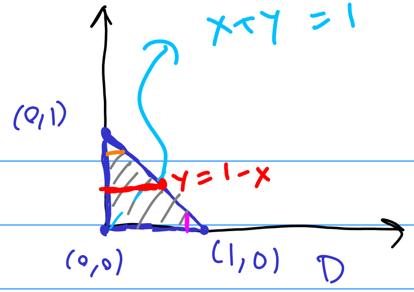
$\mathbb{E}[X] = \mathbb{E}[Y]$ (SYMMETRY)

SUBBING IN $f_{X,Y}$

$$\mathbb{E}[XY] = \iint_{\mathbb{R}^2} xy \, f_{X,Y}(x,y)\, dx\, dy = \iint_D 2xy \, dx\, dy$$

$$\mathbb{E}[X] = \iint_{\mathbb{R}^2} x \, f_{X,Y}(x,y)\, dx\, dy = \iint_D 2x \, dx\, dy$$

$$D \equiv \int \left(\int dy \right) dx \qquad [\text{TYPE I ?}]$$



$$\mathbb{E}[XY] = \iint_D 2xy \, dx\, dy = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1-x} (2xy)\, dy \right) dx$$

$$= \int_0^1 \left[x y^2 \right]_{y=0}^{y=1-x} dx = \int_0^1 x(1-x)^2 dx$$

$$= 1/12$$

llly,

$$\mathbb{E}[X] = \iint_D 2x \, dx\, dy = \int_{x=0}^{x=1} 2x \int_{y=0}^{1-x} dy = \int_0^1 2x(1-x)\, dx = \frac{1}{3}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]$$

$$= \frac{1}{12} - \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) = \frac{3-4}{36} = -\frac{1}{36}$$

$$\mathrm{Cov}(X,Y) < 0 \implies \text{-ve CORRELATED!}$$

$X, Y \rightarrow$ INDEPENDENT R.V.s.

WHAT IS $\text{Cov}(X, Y)$?

RECALL:

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y] \qquad (\because X, Y \text{ ARE INDEP.})$$

$$\Rightarrow \text{Cov}(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$$

INDEPENDENT R.V.s ARE UNCORRELATED

(INDEPENDENT $\Rightarrow$ UNCORRELATED)

WARNING: CONVERSE IS FALSE!

i.e., UNCORELLATED $\not\Rightarrow$ INDEPENDENT

Fact 8.31. Independent random variables are uncorrelated. The converse *does not hold in general.* That is, there are uncorrelated random variables that are not independent.

Example 8.32. Let X be uniform on the set $\{-1, 0, 1\}$ and $Y = X^2$.

$p_X(-1) = p_X(0) = p_X(1) = 1/3$

$Y \in \{0, 1\}$, $p_Y(0) = p_X(0) = 1/3$

$p_Y(1) = p_X(1) + p_X(-1) = 2/3$

$$\mathbb{E}[XY] = \mathbb{E}[X \cdot X^2] = \mathbb{E}[X^3]$$

CHECK,

$k \in \{-1, 0, 1\}$, $k^3 = k$

$$\therefore \mathbb{E}[XY] = \mathbb{E}[X] = (-1)\, p_X(-1) + 0 \cdot p_X(0) + 1 \cdot p_X(1)$$

$$= \frac{-1}{3} + 0 + \frac{1}{3} = 0$$

$$\mathrm{Cov}(X,Y) = \cancel{\mathbb{E}[XY]} - \cancel{\mathbb{E}[X]}\,\mathbb{E}[Y] = 0$$

BUT: X, Y ARE <u>NOT</u> INDEPENDENT

$$P(X=1, Y=0) = 0 \neq P(X=1)P(Y=0) = 1/9$$

PROPERTIES OF Cov

COMPARE TO WHAT HAPPENS WHEN X_j INDEPENDENT.

Fact 8.27. (Variance of a sum) Let $X_1, \ldots, X_n$ be random variables with finite variances and covariances. Then

CORRECTION

$$\operatorname{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \operatorname{Var}(X_i) + 2 \sum_{1 \le i < j \le n} \operatorname{Cov}(X_i, X_j). \tag{8.21}$$

(n = 2)

$$\operatorname{Var}(X + Y) = \operatorname{Var}(X) + \operatorname{Var}(Y) + 2\operatorname{Cov}(X, Y).$$

[COMPARE $(x+y)^2 = x^2 + y^2 + 2xy$]

(n = 3)

$$\operatorname{Var}(X_1 + X_2 + X_3) = \operatorname{Var}(X_1) + \operatorname{Var}(X_2) + \operatorname{Var}(X_3) + 2\operatorname{Cov}(X_1, X_2) + 2\operatorname{Cov}(X_1, X_3) + 2\operatorname{Cov}(X_2, X_3).$$

[COMPARE

$(x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

Var ≈ SQUARING ONE NUMBERS

Cov ≈ MULTIPLYING TWO NUMBERS

Pf : $(n=2)$ $\quad \mu_{X_1} = \mathbb{E}X_1,\ \mu_{X_2} = \mathbb{E}X_2, \quad \mathbb{E}(X_1+X_2) = \mu_{X_1}+\mu_{X_2}$

$$\mathrm{Var}(X_1+X_2) = \mathbb{E}\left[\left[(X_1+X_2) - (\mu_{X_1}+\mu_{X_2})\right]^2\right]$$

$$= \mathbb{E}\left[\left[(X_1-\mu_{X_1}) + (X_2-\mu_{X_2})\right]^2\right]$$

$$= \mathbb{E}\left[(X_1-\mu_{X_1})^2 + (X_2-\mu_{X_2})^2 + 2(X_1-\mu_1)\cdot(X_2-\mu_2)\right]$$

$$\overset{\text{L.O.E.}}{=} \underbrace{\mathbb{E}\left[(X_1-\mu_{X_1})^2\right]}_{\mathrm{Var}(X_1)} + \underbrace{\mathbb{E}\left[(X_2-\mu_{X_2})^2\right]}_{\mathrm{Var}(X_2)} + 2\underbrace{\mathbb{E}\left[(X_1-\mu_1)\cdot(X_2-\mu_2)\right]}_{\mathrm{Cov}(X_1,X_2)}$$

FOR LARGER n,

USE INSTEAD THE FACT THAT

IF $x_1, \ldots, x_n \in \mathbb{R}$

$$(x_1 + x_2 + \ldots + x_n)^2 = x_1^2 + x_2^2 + \ldots + x_n^2$$
$$+ 2x_1x_2 + 2x_1x_3 + \ldots + 2x_1x_n$$
$$+ 2x_2x_3 + \cdots + \cdots + 2x_{n-1}x_n$$

$$\left(\sum_j x_j\right)^2 = \left(\sum_j x_j^2\right) + \left(\sum_{i<j} 2x_ix_j\right)$$

INDEPENDENT.

Fact 8.27. (Variance of a sum) Let $X_1, \ldots, X_n$ be random variables with finite variances and covariances. Then

$$\operatorname{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \operatorname{Var}(X_i) + \cancel{2 \sum_{1 \le i < j \le n} \underbrace{\operatorname{Cov}(X_i, X_j)}_{=0}}. \tag{8.21}$$

RECALL: IND ⇒ UNCORR.

$$\therefore \operatorname{Var}\left(\sum X_i\right) = \sum_i \operatorname{Var}(X_i)$$

ALREADY KNOW THIS!

NOTE: ALL WE NEEDED,

$$\mathrm{Cov}(X_i, X_j) = 0 \quad \text{IF} \quad i \neq j$$

Fact 8.28. (Variance of a sum of uncorrelated random variables) Let $X_1, \ldots, X_n$ be uncorrelated random variables with finite variances. Then

$$\mathrm{Var}(X_1 + X_2 + \cdots + X_n) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + \cdots + \mathrm{Var}(X_n).$$

BREAK TILL
10:10 AM

Fact 8.33. The following statements hold when the covariances are well defined.

SYMMETRY $\leftarrow$ (i) $\mathrm{Cov}(X, Y) = \mathrm{Cov}(Y, X)$.

(ii) $\mathrm{Cov}(aX + b, Y) = a\,\mathrm{Cov}(X, Y)$ for any real numbers a, b.

(iii) For random variables X_i and Y_j and real numbers a_i and b_j,

BILINEARITY (SEE HW 8)

$$\mathrm{Cov}\left(\sum_{i=1}^{m} a_i X_i, \sum_{j=1}^{n} b_j Y_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_i b_j \,\mathrm{Cov}(X_i, Y_j). \tag{8.25}$$

RECALL: $\mathrm{Var}[aX+b] = \mathrm{Cov}(aX+b, aX+b)$

[(ii) w/ $Y \mapsto aX+b$] $= a\,\mathrm{Cov}(X, aX+b) \overset{(i)}{=} a\,\mathrm{Cov}(aX+b, X)$

[(ii)] $= a\left[a\,\mathrm{Cov}(X,X)\right] = a^2\,\mathrm{Var}[X]$

Pf of (i)

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$= \mathbb{E}[YX] - \mathbb{E}[Y]\,\mathbb{E}[X] = \mathrm{Cov}(Y,X)$$

Pf of (ii)

$$\mathrm{Cov}(aX+b, Y) = \mathbb{E}\big((aX+b)\cdot Y\big) - \mathbb{E}(aX+b)\,\mathbb{E}[Y]$$

$$\overset{\text{LoE}}{=} \mathbb{E}[aXY + bY] - [a\mathbb{E}X + b]\cdot \mathbb{E}Y$$

$$\overset{\text{LoE}}{=} a\mathbb{E}[XY] + \cancel{b\,\mathbb{E}[Y]} - a\,\mathbb{E}[X]\,\mathbb{E}[Y] - \cancel{b\,\mathbb{E}[Y]}$$

$$= a\big[\,\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]\,\big] = a\,\mathrm{Cov}(X,Y)$$

Pf of (iii) for small cases

$m = 1, n = 1$

$$\mathrm{Cov}(a_1 X_1, b_1 Y_1) \overset{(ii)}{=} a_1 b_1 \mathrm{Cov}(X_1, Y_1)$$

$m = 2, n = 1$

$$\mathrm{Cov}(a_1 X_1 + a_2 X_2, b_1 Y_1) = \mathbb{E}\left[(a_1 X_1 + a_2 X_2) \cdot b_1 Y_1\right] - \mathbb{E}\left[a_1 X_1 + a_2 X_2\right] \cdot \mathbb{E}[b_1 Y_1$$

$$= a_1 b_1 \mathbb{E}(X_1 Y_1) + a_2 b_1 \mathbb{E}(X_2 Y_1) - a_1 b_1 \mathbb{E}(X_1) \mathbb{E}(Y_1) - a_2 b_1 \mathbb{E}(X_2) \mathbb{E}(Y_1)$$

$$= a_1 b_1 \mathrm{Cov}(X_1, Y_1) + a_2 b_1 \mathrm{Cov}(X_2, Y_1)$$

SIMILAR PROOF

$$\operatorname{Cov}\left(\sum_{i=1}^{m} a_i X_i, Y\right) = \sum_{i=1}^{m} a_i \operatorname{Cov}(X_i, Y) \quad \text{— } (A)$$

$$\therefore \operatorname{Cov}\left(\sum_{i=1}^{m} a_i X_i, \underbrace{\sum_{j=1}^{n} b_j Y_j}_{\text{FIXED, USE } (A)}\right) = \sum_{i=1}^{m} a_i \operatorname{Cov}\left(\underbrace{X_i}_{\text{FIXED, USE } (A)\text{, SYMMETRY}}, \sum_{j=1}^{n} b_j Y_j\right)$$

$$= \sum_{i=1}^{m} a_i \sum_{j=1}^{n} b_j \operatorname{Cov}(X_i, Y_j)$$

DRAWBACK OF COVARIANCE : SENSITIVE TO LARGE FLUCTUATIONS IN EITHER X OR Y.

$$\mathbb{E}\left(\left(X-\mu_X\right)\cdot\left(Y-\mu_Y\right)\right)$$

SOLUTION : NORMALIZE

Definition 8.35. The **correlation** (or **correlation coefficient**) of two random variables X and Y with positive finite variances is defined by

$$\operatorname{Corr}(X,Y)=\frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)}\sqrt{\operatorname{Var}(Y)}}. \tag{8.27}$$

The correlation coefficient is sometimes denoted by $\rho(X,Y)$ or simply by ρ.

NOTE :

RECALL, $\tilde{X} = \frac{X - \mu_X}{\sigma_X} = \frac{X - \mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}}$

$$\mathbb{E}(\tilde{X}) = \mathbb{E}\left[\frac{X - \mu_X}{\sigma_X}\right] = \frac{\mathbb{E}(X) - \mu_X}{\sigma_X} = 0$$

$$\mathrm{Var}(\tilde{X}) = \mathrm{Var}\left[\frac{X - \mu_X}{\sigma_X}\right] = \frac{1}{\sigma_X^2}\mathrm{Var}(X) = 1$$

AFFINE

MEASURE OF CORRELATION

$$\mathbb{E}(\tilde{X}\tilde{Y}) = \mathbb{E}\left[\left(\frac{X - \mu_X}{\sigma_X}\right) \cdot \left(\frac{Y - \mu_Y}{\sigma_Y}\right)\right]$$

$$\mathbb{E}(\tilde{X}\tilde{Y}) = \mathbb{E}\left[\frac{XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y}{\sigma_X\sigma_Y}\right]$$

$$= \frac{1}{\sigma_X\sigma_Y}\left[\mathbb{E}[XY] - \underbrace{\mu_X\mathbb{E}[Y]}_{\mu_X\mu_Y} - \underbrace{\mu_Y\mathbb{E}[X]}_{\mu_X\mu_Y} + \mu_X\mu_Y\right]$$

$$= \frac{1}{\sigma_X\sigma_Y}\left[\mathbb{E}[XY] - \mu_X\mu_Y\right] = \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

CORRELATION

Theorem 8.36. *Let X and Y be two random variables with positive finite variances. Then we have these properties.*

(a) *Let a, b be real numbers with $a \neq 0$. Then* $\operatorname{Corr}(aX+b, Y) = \boxed{\frac{a}{|a|}}\operatorname{Corr}(X,Y)$. → ONLY CHANGES SIGN !

(b) $-1 \leq \operatorname{Corr}(X,Y) \leq 1$.

(c) $\operatorname{Corr}(X,Y) = 1$ *if and only if there exist* $a > 0$ *and* $b \in \mathbb{R}$ *such that* $Y = aX + b$.

(d) $\operatorname{Corr}(X,Y) = -1$ *if and only if there exist* $a < 0$ *and* $b \in \mathbb{R}$ *such that* $Y = aX + b$.

RECALL : $\operatorname{Var}(X) = 0 \iff X = \mu_X$ w/ PROB $= 1$

RECALL $\rho \equiv \text{Corr}$

Pf OF (a)

$$\rho(aX+b, Y) = \frac{\text{Cov}(aX+b, Y)}{\sqrt{\text{Var}(aX+b)} \cdot \sqrt{\text{Var}(Y)}}$$

$$= \frac{a\,\text{Cov}(X,Y)}{\sqrt{a^2\,\text{Var}(X)} \cdot \sqrt{\text{Var}(Y)}}$$

[PROP. OF Cov / Var W/ AFFINE]

$(\sqrt{a^2} = |a|)$

$$= \frac{a}{|a|} \cdot \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}\,X}\,\sqrt{\text{Var}\,Y}}$$

$\rho(X,Y)$

Pf of (b)

RECALL : $\rho(X,Y) = \mathbb{E}(\tilde{X}\tilde{Y})$

CONSIDER, $\mathbb{E}\left[(\tilde{X} - \tilde{Y})^2\right] \geq 0 \quad \left[\because (\tilde{X} - \tilde{Y})^2 \geq 0\right]$

O.T.O.H, $\mathbb{E}\left[\tilde{X}^2 + \tilde{Y}^2 - 2\tilde{X}\tilde{Y}\right] = \underbrace{\mathbb{E}[\tilde{X}^2]}_{1} + \underbrace{\mathbb{E}[\tilde{Y}^2]}_{1} - 2\underbrace{\mathbb{E}[\tilde{X}\tilde{Y}]}_{\rho(\tilde{X},\tilde{Y})}$

$$2 - 2\,\mathbb{E}(\tilde{X}\tilde{Y}) \geq 0 \Rightarrow \mathbb{E}(\tilde{X}\tilde{Y}) \leq 1 \Rightarrow \rho(X,Y) \leq 1$$

TO PROVE $\rho(X,Y) \geq -1$,

USE INSTEAD

$$\mathbb{E}\left(\left(\tilde{X}+\tilde{Y}\right)^2\right) \geq 0$$

PF OF (C) [(d) IS SIMILAR]

NOTE :
$$\rho(X,X) = \frac{Cov(X,X)}{\left[\sqrt{Var(X)}\right]^2} = \frac{Var(X)}{Var(X)} = 1$$

$$\rho(X, aX+b) = \frac{a}{|a|}\rho(X,X) = \frac{a}{|a|} = 1 \quad (\text{IF } a>0)$$

$$Y = aX + b, \quad a > 0 \Rightarrow \rho(X,Y) = 1$$

LET'S PROVE CONVERSE.

$$\therefore \rho(X,Y) = 1$$

RECALL:

$$0 \leq \mathbb{E}[(\tilde{X} - \tilde{Y})^2] = 2 - 2\rho(X,Y) = 2 - 2 \cdot 1 = 0$$

$$\Rightarrow \mathbb{E}[(\tilde{X} - \tilde{Y})^2] = 0$$

$$\Rightarrow \tilde{X} - \tilde{Y} = 0 \quad \text{w/ PROB} = 1$$

$$\Rightarrow \tilde{X} = \tilde{Y} \Rightarrow \frac{X - \mu_X}{\sigma_X} = \frac{Y - \mu_Y}{\sigma_Y} \Rightarrow Y = aX + b \quad (\text{w/ PROB} = 1)$$

§ 7.1 SUMS OF INDEPENDENT R.V.s.

RECALL: $X_1, X_2 \to$ INDEPENDENT DIE ROLLS

$$S = X_1 + X_2$$

$X_1, X_2 \in \{1, \dots, 6\}$ $\qquad p_{X_j}(1) = p_{X_j}(2) = \cdots = p_{X_j}(6) = \frac{1}{6}$

p.m.f. OF S? $\qquad \{S=7\} = \{X_1=1, X_2=6\} \cup \{X_1=2, X_2=5\} \cup \cdots \cup \{X_1=6, X_2=1\}$

e.g. $p_S(7) = P(S=7) = P(X_1=1, X_2=6) + P(X_1=2, X_2=5) + \cdots + P(X_1=6, X_2=1)$

$$= p_{X_1}(1)\, p_{X_2}(6) + \cdots + p_{X_1}(6)\, p_{X_2}(1)$$

e.g. $p_S(7) = p_{X_1}(1)\, p_{X_2}(6) + \cdots + p_{X_1}(6)\, p_{X_2}(1)$

$$= \left(\frac{1}{6} \cdot \frac{1}{6}\right) \times 6 = \frac{1}{6}$$

NOTE:

$$p_S(7) = \sum_{s+t=7} p_{X_1}(s)\, p_{X_2}(t)$$

$[s \to X_1,\ t \to X_2]$

$$= \sum_{t=1}^{7} p_{X_1}(7-t)\, p_{X_2}(t)$$

IN GENERAL, IF X, Y DISCRETE & INDEPENDENT, (BUT NOT NEC. i.i.d.)

$S = X + Y$,

THEN $\underbrace{P(S = n)}_{p_S(n)} = P\left[\bigcup_{\substack{k, l \\ k + l = n}} \{ X = k, Y = l \} \right]$

$$= \sum_{\substack{k, l \\ k + l = n}} P(X = k, Y = l) \quad \rightarrow P(X=k) \cdot P(Y=l), \text{ IND.}$$

$$= \sum_{k+l=n} p_X(k)\, p_Y(l)$$

$$P_S(n) = \sum_{k+l=n} P_X(k)\, P_Y(l)$$

$$= \sum_{k:\ X \text{ TAKES VALUE } k} P_X(k)\, P_Y(n-k)$$

$$= \sum_{l:\ Y \text{ TAKES VALUE } l} P_X(n-l)\, P_Y(l)$$

CONVOLUTION OF P_X & P_Y

$= P_X * P_Y$

P.M.F. OF SUMS OF IND. R.Vs IS THE CONVOLUTION OF THEIR P.M.F.S

Fact 7.1. (Convolution of distributions) If X and Y are independent discrete random variables with probability mass functions p_X and p_Y, then the probability mass function of $X + Y$ is

$$p_{X+Y}(n) = p_X * p_Y(n) = \sum_k p_X(k)\, p_Y(n-k) = \sum_\ell p_X(n-\ell)\, p_Y(\ell). \tag{7.2}$$

$$\sum_{k+\ell=n} p_X(k)p_Y(\ell)$$

If X and Y are independent continuous random variables with density functions f_X and f_Y then the density function of $X + Y$ is

$$f_{X+Y}(z) = f_X * f_Y(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\, dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\, dy. \tag{7.3}$$

CHANGE OF VARIABLE

Y = Z − X

$$\underbrace{F_{X+Y}}_{\text{c.d.f.}}(z) = \int_{-\infty}^{z} f_{X+Y}(w)\,dw \qquad \text{(RECALL)}$$

O.T.O.H.

$$F_{X+Y}(z) = P(X+Y \leq z)$$

$$= \iint_{\{x+y \leq z\}} f_{X,Y}(x,y)\,dx\,dy \overset{\text{(IND.)}}{=} \iint_{\{x+y \leq z\}} f_X(x)\,f_Y(y)\,dx\,dy$$

$\{x+y \leq z\} \to$ BOTH TYPE I & TYPE II.

$$\therefore F_{X+Y}(z) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-x} f_X(x)\, f_Y(y)\, dy \right] dx$$

SET $w = y + x$ INSIDE $(y = w - x)$

$$F_{X+Y}(z) = \underbrace{\int_{-\infty}^{\infty} \left[\int_{-\infty}^{z} \right.}_{\text{RECTANGULAR} \to \text{FUBINI}} \left. f_X(x)\, f_Y(w-x)\, dw \right] dx$$

$$\therefore F_{X+Y}(z) = \int_{-\infty}^{z} \Bigg[\underbrace{\int_{-\infty}^{\infty} f_X(x)\, f_Y(w-x)\, dx}_{f_X * f_Y (w)} \Bigg] dw$$

$$F_{X+Y}(z) = \int_{-\infty}^{z} \boxed{f_{X+Y}(w)}\, dw$$

$$f_{X+Y}(w) = (f_X * f_Y)(w)$$

Example 7.2. (Convolution of Poisson random variables) Suppose that $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ and these are independent. Find the distribution of $X + Y$.

$$X, Y \in \{0, 1, 2, 3, \dots\}$$

$$P_X(k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$P_Y(\ell) = e^{-\mu} \cdot \frac{\mu^\ell}{\ell!}$$

$$p_{X+Y}(n) = (p_X * p_Y)(n)$$

$$= \sum_{\substack{k+l=n \\ k,l \in \{0,1,2,\dots\}}} p_X(k)\, p_Y(l)$$

NOTE: $0 \le k \le n$

$$p_{X+Y}(n) = \sum_{k=0}^{n} p_X(k)\, p_Y(n-k)$$

$$p_{X+Y}(n) = \sum_{k=0}^{n} p_X(k)\, p_Y(n-k)$$

$$= \sum_{k=0}^{n} \left(e^{-\lambda} \frac{\lambda^k}{k!}\right)\left(e^{-\mu} \cdot \frac{\mu^{(n-k)}}{(n-k)!}\right)$$

$$= e^{-(\lambda+\mu)} \boxed{\sum_{k=0}^{n} \frac{1}{k!\,(n-k)!} \lambda^k \mu^{n-k}}$$

↕ DIFF IS $n!$

RECALL:

$$(\lambda+\mu)^n = \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mu^{n-k} = \boxed{\sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \lambda^k \mu^{n-k}}$$

$$\therefore \quad P_{X+Y}(n) = \frac{e^{-(\lambda+\mu)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!} \lambda^k \mu^{n-k}$$

$$= \boxed{\frac{e^{-(\lambda+\mu)}}{n!} (\lambda+\mu)^n} \rightarrow \text{p.m.f. of } Pois(\lambda+\mu)$$

$\lambda+\mu \rightarrow \lambda$, $n \rightarrow k$

$$e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$\therefore X + Y \sim \text{Pois}(\lambda + \mu)$$

SUMS OF INDEPENDENT POISSON VARIABLES ARE POISSON W/ MEAN = SUM OF MEANS