MATH 201 (SUMMER 2023, SESH A2)

LECTURE 16: 06/13/23

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. OFFICE HOURS : W : 11:15 AM – 12:15 PM (OR BY APPT.)
2. UPCOMING DEADLINES : (i) HW 7 – TODAY , (iii) HW 8 – THURS
   (ii) WW 8 – WED (iv) WW 9 – SAT
   NEXT WEEK'S DEADLINES ARE ALSO UP! [2 HW, 1 WW]
3. GRADING POLICY : EXTRA CREDIT – HW 10
   CAN INCREASE UP TO 5% OF MAX.
4. CLASS ON T, JUNE 20th TO BE FLIPPED.
5. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§7.1 SUMS OF INDEPENDENT R.Vs. (CONTD.)

RECALL :

① $\boxed{X, Y \text{ DISCRETE \& INDEPENDENT}} \Rightarrow \boxed{X+Y \text{ DISCRETE, \& } p_{X+Y} = p_X * p_Y}$

$$\therefore \; p_{X+Y}(n) = \sum_{k+l=n} p_X(k) p_Y(l) = \sum_k p_X(k) p_Y(n-k) = \sum_l p_X(n-l) p_Y(l)$$

② $\boxed{X, Y \text{ CONT. \& IND.}} \Rightarrow \boxed{X+Y \text{ CONTINUOUS, } f_{X+Y} = f_X * f_Y}$

$$\therefore \; f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y)\,dy$$

RECALL :

$$X \sim Pois(\lambda)$$

$$Y \sim Pois(\mu)$$

$$X, Y \quad \text{IND.}$$

$$\Rightarrow X + Y \sim Pois(\lambda + \mu)$$

Example 7.4. (Convolution of binomials with the same success probability) Let $X \sim \text{Bin}(m_1, p)$ and $Y \sim \text{Bin}(m_2, p)$ be independent. Find the distribution of $X + Y$.

$$P(X=k) = \binom{m_1}{k} p^k (1-p)^{m_1-k}$$

FACT :

$$\sum_{k+l} \binom{m_1}{k}\binom{m_2}{l} = \binom{m_1+m_2}{n}$$

$$P(Y=l) = \binom{m_2}{l} p^l (1-p)^{m_2-l}$$

$$P(X+Y=n) = \sum_{k+l=n} P(X=k)\,P(Y=l) = \sum_{k+l=n} \binom{m_1}{k}\binom{m_2}{l} p^{\overset{n}{\widetilde{k+l}}} (1-p)^{m_1+m_2-\overset{n}{\widetilde{k-l}}}$$

$$= \boxed{\sum_{k+l=n} \binom{m_1}{k}\binom{m_2}{l}} p^n (1-p)^{m_1+m_2-n}$$

$$P(X+Y=n) = \underbrace{\binom{m_1+m_2}{n} p^n \; (1-p)^{m_1+m_2-n}}$$

$$\Rightarrow X+Y \sim Bin(m_1+m_2, p)$$

$$X : \; \underbrace{\_\;\_\;\_\;\_\;\_\;\_}_{m_1}\;\_\;\_$$

$$Y : \; \_\;\underbrace{\_\;\_\;\_}_{m_2}\;\_\;\_\;\_\;\_\;\_$$

m_1+m_2 IND. BERNOULLI TRIALS.

$\Rightarrow \quad X + Y \sim \text{Bin}(m_1 + m_2, p)$ [W/ NO CALC.]

$\therefore$ SUMS OF $\text{Bin}(-, p)$ ARE $\text{Bin}(-, p)$ AND # OF TRIALS ADDED.

Example 7.5. (Convolution of geometric random variables) Let X and Y be independent geometric random variables with the same success parameter $p < 1$. Find the distribution of $X + Y$.

$X, Y \sim \text{Geom}(p)$

$P(X = k) = P(Y = k) = (1-p)^{k-1} p$ $\quad (k \in \{1, 2, 3, \ldots\})$

$(n \in \{2, 3, 4, \ldots\})$

$$P(X+Y=n) = \sum_k P(X=k)\, P(Y=n-k)$$

$k \geq 1, \; n-k \geq 1 \Rightarrow 1 \leq k \leq n-1$

$$P(X+Y=n) = \sum_{k=1}^{n-1} \left[(1-p)^{k-1} \cdot p \right] \cdot \left[(1-p)^{n-k-1} \cdot p \right]$$

$(1-p)^{n-2}$

$$P(X+Y=n) = \sum_{k=1}^{n-1} \left[(1-p)^{\boxed{k-1}} \cdot p \right] \cdot \left[(1-p)^{\boxed{n-k-1}} \cdot p \right]$$

$$= \sum_{k=1}^{n-1} (1-p)^{n-2} \cdot p^2 = (n-1)(1-p)^{n-2} p^2$$

$$p_{X+Y}(n) = P(X+Y=n) = \underbrace{(n-1)(1-p)^{n-2} \cdot p^2}$$

$$\underbrace{\underline{F}\ \ \underline{F}\ \ \underline{F}\ \ \underline{S}}_{X}\ \ \underbrace{\underline{F}\ \ \underline{F}\ \ \underline{S}}_{Y}$$

N = # OF CONSECUTIVE, IND. BERNOULLI TRIALS, UNTIL WE SEE 2 SUCCESSES.

$$N = X + Y, \qquad X, Y \sim \text{Geom}(p)$$

NEED TO CHOOSE POS. OF FIRST SUCCESS.

1 2 . . . . n

F F F S F F $\boxed{S}$

$$P(N=n) = (n-1)\, p^2 (1-p)^{n-2}$$

AGREES w/ CONVOLUTION!

$$X = \underbrace{X_1 + X_2 + \cdots + X_k}_{k \text{ CONVOLUTIONS !!!}} \qquad X_j \sim \text{Geom}(p), \text{ i.i.d.}$$

SAME IDEA?

$\hookrightarrow$ REINTERPRET X AS THE WAITING FLIPS (i.e. # OF BERNOULLI TRIALS) UNTIL WE SEE k SUCCESSES.

$$\underbrace{\_\ \ \_\ \ \_\ \ \underline{S}}_{X_1}\ \ \underbrace{\_\ \ \_\ \ \underline{S}}_{X_2}\ \ \cdots\cdots\ \ \underbrace{\_\ \ \underline{S}}_{X_k}$$

1 2 3 4 . . . . . . n

$$\underbrace{\underline{F}\ \underline{F}\ \underline{F}\ \underline{S}}_{X_1}\ \ \underbrace{\underline{F}\ \underline{F}\ \underline{S}}_{X_2}\ \cdots\ \underbrace{\underline{F}\ \ \boxed{S}}_{X_k}$$

NEED TO FIX POSITION OF OTHER $k-1$ SUCCESSES

$$P(X = n) = \binom{n-1}{k-1} p^{k} (1-p)^{n-k}$$

$$\therefore \quad P_{X_1+X_2+\cdots+X_k}(n) = \binom{n-1}{k-1} p^{k} (1-p)^{n-k}$$

$$n \in \{k, k+1, k+2, \ldots\}$$

Definition 7.6. Let k be a positive integer and $0 < p < 1$. A random variable X has the **negative binomial distribution** with parameters (k, p) if the set of possible values of X is the set of integers $\{k, k+1, k+2, \dots\}$ and the probability mass function is

$$P(X = n) = \binom{n-1}{k-1} p^k (1-p)^{n-k} \qquad \text{for } n \geq k.$$

Abbreviate this by $X \sim \text{Negbin}(k, p)$.

Note that the Negbin$(1, p)$ distribution is the same as the Geom(p) distribution.

$\text{Negbin}(k,p) \simeq \boxed{k \ \text{Geom}(p)}$

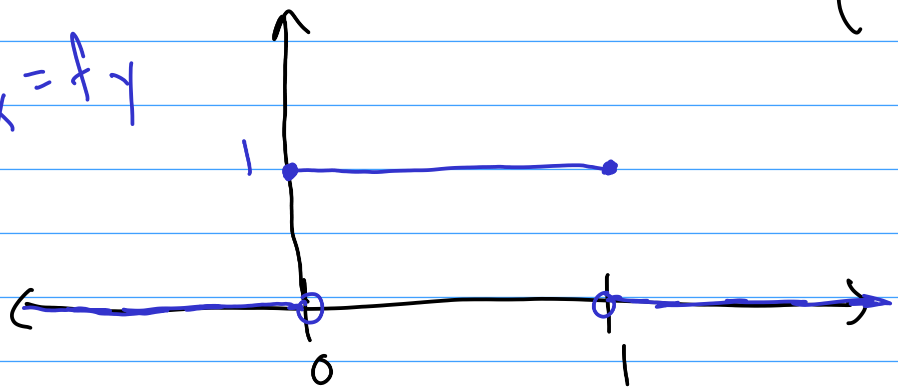
INDEPENDENT.

Example 7.13. (Convolution of uniform random variables) Suppose that X and Y are independent and distributed as Unif[0, 1]. Find the distribution of $X+Y$.

$$X, Y \sim \text{Unif}[0,1]$$

$$f_X(t) = f_Y(t) = \begin{cases} 1 & \text{IF } t \in [0,1] \\ 0 & \text{o.w.} \end{cases}$$

$f_X = f_Y$



$$f_{X+Y}(z) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{x \in [0,1]} \underbrace{f_Y(z-x)}_{z-x \in [0,1]} dx$$

$$\left.\begin{aligned} 0 &\le x \le 1 \\ 0 &\le z - x \le 1 \end{aligned}\right\} \Rightarrow \max(0, z-1) \le x \le \min(1, z)$$

CASES :

① $z > 2 \Rightarrow z - 1 > 1 \Rightarrow x > 1$ ($\#$ $x \le 1$)

$\Rightarrow$ NO x SATISFYING BOTH INEQ. $f_{x+y}(z) = 0$

② $z < 0 \Rightarrow x < 0$ ($\#$ $x > 0$)

$\Rightarrow$ |||ly $f_{x+y}(z) = 0$

$$\max(0, z-1) \le x \le \min(1, z)$$

(3) $0 \le z \le 1$:

- $z - 1 \le 0 \Rightarrow \max(0, z-1) = 0$

  & $\min(1, z) = z$

<u>INEQ.</u> : $0 \le x \le z$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx = \int_0^z dx = z$$

$$(0 \le z \le 1)$$

$$\max(0, z-1) \le x \le \min(1, z)$$

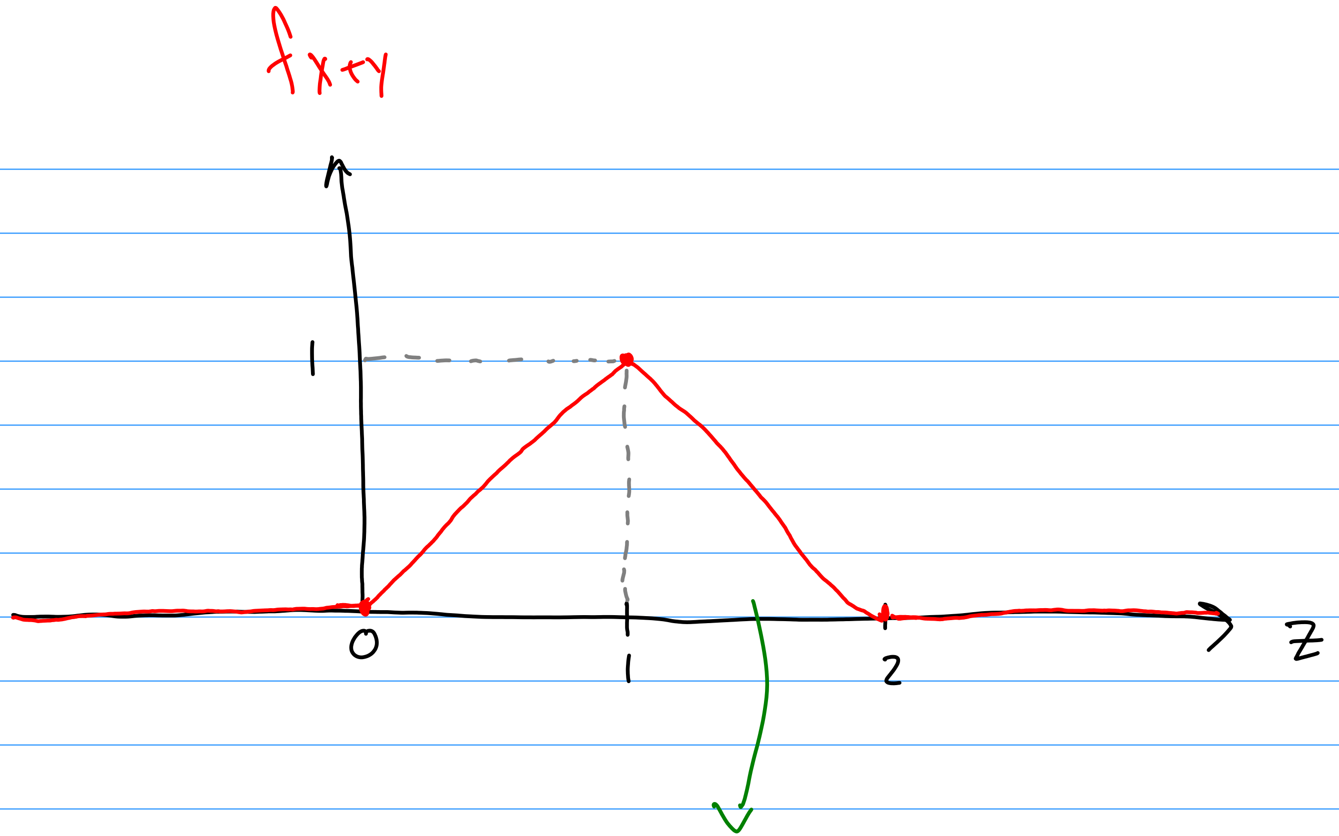
(3) $1 \le z \le 2$: $z - 1 \geq 0 \Rightarrow \max(0, z-1) = z-1$

$\min(1, z) = 1$

INEQ. : $z-1 \le x \le 1$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx = \int_{z-1}^{1} dx = 2 - z$$

$(1 \le z \le 2)$



$$\text{AREA} = \frac{1}{2} \cdot (2) \cdot (1) = 1$$

Fact 7.9. Assume $X_1, X_2, \ldots, X_n$ are independent random variables with $X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, $a_i \neq 0$, and $b \in \mathbb{R}$. Let $X = a_1X_1 + \cdots + a_nX_n + b$. Then $X \sim \mathcal{N}(\mu, \sigma^2)$ where

$$\mu = a_1\mu_1 + \cdots + a_n\mu_n + b \quad \text{and} \quad \sigma^2 = a_1^2\sigma_1^2 + \cdots + a_n^2\sigma_n^2.$$

SUMS OF INDEPENDENT NORMALS ARE NORMAL, WITH MEAN & VARIANCE PARAMS ADDED.

SIMPLEST CASE $\quad X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$ (IND.)

$X + Y \sim N(\mu, \sigma^2) \Rightarrow$ LoE $\quad \mu = \mu_1 + \mu_2$, $\sigma^2 = \sigma_1^2 + \sigma_2^2$

$$N(\mu_1, \sigma_1^2) + N(\mu_2, \sigma_2^2) + \cdots + N(\mu_k, \sigma_k^2)$$

$$\| \quad \text{(IND.)}$$

$$N(\mu_1 + \mu_2 + \cdots + \mu_k,\ \sigma_1^2 + \sigma_2^2 + \cdots + \sigma_k^2)$$

Example 7.10. Let $X \sim \mathcal{N}(1,3)$ and $Y \sim \mathcal{N}(0,4)$ be independent and let $W = \frac{1}{2}X - Y + 6$. Identify the distribution of W.

$$W = \underbrace{\frac{1}{2}X}_{\sim \mathcal{N}(1/2,\, 3/4)} - \underbrace{Y}_{\sim \mathcal{N}(0,4)} + \underbrace{6}_{\sim \mathcal{N}(6,0)}$$

$$\Rightarrow \quad W \sim \mathcal{N}\left(\frac{1}{2} + 0 + 6,\ \frac{3}{4} + 4\right) = \mathcal{N}\left(\frac{13}{2},\ \frac{19}{4}\right)$$

Example 7.12. (Convolution of exponential random variables) Suppose that X and Y are independent ~~Exp(λ)~~ random variables. Find the density of $X + Y$.

$$X \sim \mathrm{Exp}(\lambda) \quad , \quad Y \sim \mathrm{Exp}(\mu)$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$f_Y(y) = \begin{cases} \mu e^{-\mu y} & y \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

$$f_{X+Y}(z) = (f_X * f_Y)(z)$$

$$= \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{x \geq 0} \; \underbrace{f_Y(z-x)}_{z - x \geq 0} \, dx$$

$\Rightarrow \quad 0 \leq x \leq z$ (FOR INTEGRAND TO NOT VANISH)

$f_{X+Y}(z) = 0$ IF $z < 0$

ASSUME $z \geq 0$,

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_0^z \left(\lambda e^{-\lambda x}\right)\left(\mu e^{-\mu(z-x)}\right) dx$$

$$= \lambda \mu e^{-\mu z} \int_0^z e^{(\mu-\lambda)x}\, dx$$

$$\int_0^z e^{(\mu-\lambda)x}\,dx$$

I $\quad \lambda = \mu$

$$\int_0^z e^{0x}\,dx = \int_0^z dx = z$$

II $\quad \lambda \neq \mu$

$$\int_0^z e^{(\mu-\lambda)x}\,dx = \frac{1}{\boxed{\mu-\lambda}} e^{(\mu-\lambda)x}\Bigg]_{x=0}^{x=z} = \frac{e^{(\mu-\lambda)z}-1}{\mu-\lambda}$$

(NON-ZERO

$$f_{X+Y}(z) = \lambda\mu \boxed{e^{-\mu z}} \boxed{\int_0^z e^{(\mu-\lambda)x}\,dx}$$

$\therefore$ If $\mu = \lambda$, $\quad f_{X+Y}(z) = \lambda^2 z e^{-\lambda z} \quad (z>0)$

If $\mu \neq \lambda$, $\quad f_{X+Y}(z) = \dfrac{\lambda\mu}{\mu-\lambda}\left(e^{-\lambda z} - e^{-\mu z}\right)$

IF $\mu = \lambda$,

$$f_{X+Y}(z) = \begin{cases} \lambda^2 z e^{-\lambda z} & , \quad z > 0 \\ 0 & , \quad o.w. \end{cases}$$

Gamma $(2, \lambda)$

IF $\mu \neq \lambda$

$$f_{X+Y}(z) = \begin{cases} \frac{\lambda\mu}{\mu - \lambda} \left(e^{-\lambda z} - e^{-\mu z} \right) & , \quad z > 0 \\ 0 & , \quad o.w. \end{cases}$$

	X	Y	$X + Y$ (ASSUMING IND.)
V. IMP. ①	$X \sim \text{Pois}(\lambda)$	$Y \sim \text{Pois}(\mu)$	$X + Y \sim \text{Pois}(\lambda + \mu)$
②	$X \sim N(\mu_1, \sigma_1^2)$	$Y \sim N(\mu_2, \sigma_2^2)$	$X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$
IMP. ③	$X \sim \text{Geom}(p)$	$Y \sim \text{Geom}(p)$	$X + Y \sim \text{NegBin}(2, p)$
NOT IMP. ④	$X \sim \text{Exp}(\lambda)$	$Y \sim \text{Exp}(\lambda)$	$X + Y \sim \text{Gamma}(2, \lambda)$

BREAK

TILL 10:25

§ 5.1 MOMENT GENERATING FUNCTION

$M_X(t)$

mgf.

Definition 5.1. The moment generating function of a random variable X is defined by $M(t) = E(e^{tX})$. It is a function of the real variable t.

e.g. DISCRETE $\rightarrow$ $M(t) = E(e^{tX}) = \sum_k e^{tk} p_X(k)$

$t \in \mathbb{R}$

CONT. $\rightarrow$ $M(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

INTUITION: FOURIER / LAPLACE TRANSFORM

Example 5.2. Let X be a discrete random variable with probability mass function

$$P(X=-1)=\tfrac{1}{3},\quad P(X=4)=\tfrac{1}{6},\quad \text{and}\quad P(X=9)=\tfrac{1}{2}.$$

Find the moment generating function of X.

$X \in \{-1, 4, 9\}$

$$M(t) = \mathbb{E}(e^{tX}) = e^{-t} P(X=-1) + e^{4t} P(X=4) + e^{9t} P(X=9)$$

$$\boxed{M(t) = \frac{e^{-t}}{3} + \frac{e^{4t}}{6} + \frac{e^{9t}}{2}}$$

Example 5.4. (Moment generating function of the Poisson distribution) Let $X \sim \text{Poisson}(\lambda)$. The calculation is an application of formula (3.24) with the Poisson probability mass function:

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!} \qquad \left(k \in \{0, 1, 2, 3, \dots\} \right)$$

$$M(t) = E(e^{tX}) = \sum_{k=0}^{\infty} \left(e^{tk}\right) \left[e^{-\lambda} \frac{\lambda^k}{k!} \right]$$

$$= e^{-\lambda} \underbrace{\boxed{\sum_{k=0}^{\infty} \frac{(e^t \lambda)^k}{k!}}}_{\exp(e^t \lambda)} = e^{-\lambda} \cdot e^{(e^t \cdot \lambda)} = \boxed{e^{\lambda(e^t - 1)}}$$

Example 5.5. (Moment generating function of the normal distribution) Let $Z \sim \mathcal{N}(0,1)$. ~~Now the computation uses formula (3.25). To evaluate the integral we~~

$$M(t) = \mathbb{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} \left(e^{tx}\right) \cdot \left(\frac{e^{-x^2/2}}{\sqrt{2\pi}}\right) dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx - x^2/2} \, dx$$

COMPLETE THE SQUARE

$$x^2 - 2tx = \underbrace{x^2 - 2tx + t^2}_{(x-t)^2} - t^2 = (x-t)^2 - t^2$$

$$-\frac{x^2}{2} + tx = -\frac{1}{2}\left[x^2 - 2tx\right]$$

$$= -\frac{1}{2}\left[(x-t)^2 - t^2\right]$$

$$M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-t)^2 - t^2\right]}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} \cdot \boxed{e^{t^2/2}}\, dx$$

$$M(t) = e^{t^2/2} \cdot \left[\underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2} \, dx}_{=1} \right]$$

$$\left(\int_{-\infty}^{\infty} P_N(x) \, dx \qquad N \sim \mathcal{N}(t, 1) \right)$$

ALTERNATIVELY, $u = x - t \Rightarrow \begin{cases} x = \infty, & u = \infty \\ x = -\infty, & u = -\infty \end{cases}$, $dx = du$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du = \Phi(\infty) = 1$$

$\therefore M(t) = e^{t^2/2}$ (for $N(0,1)$)

Example 5.6. (Moment generating function of the exponential distribution) Let $X \sim \text{Exp}(\lambda)$. Then

$$M(t) = \mathbb{E}\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$M(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$M(t) = \int_0^\infty e^{tx} \lambda e^{-\lambda x} \, dx$$

CAREFUL: (IMPROPER)

$$\int_0^N e^{tx} \lambda e^{-\lambda x} \, dx = \lambda \int_0^N e^{(t-\lambda)x} \, dx$$

CASE I : $t = \lambda$

$$M(\lambda) = \lim_{N \to \infty} \left[\lambda \int_0^N e^{(\lambda - \lambda)x} \, dx \right] = \lim_{N \to \infty} \lambda \int_0^N dx = \lim_{N \to \infty} (\lambda N)$$

$$= \infty$$

CASE II : $t > \lambda$

$$M(t) = \lim_{N\to\infty} \left[\lambda \int_0^N e^{(t-\lambda)x}\, dx \right] = \lim_{N\to\infty} \left[\lambda \, \frac{e^{(t-\lambda)x}}{(t-\lambda)} \right]_{x=0}^{x=N}$$

$$= \lim_{N\to\infty} \left[\frac{\lambda}{t-\lambda} \left(e^{(t-\lambda)N} - 1 \right) \right] = \infty$$

$t-\lambda > 0$, $\quad e^{(+ve)N} \longrightarrow \infty$

CASE III : $t < \lambda$

$$M(t) = \lim_{N \to \infty} \left[\frac{\lambda}{t-\lambda} \left(e^{(t-\lambda)N} - 1 \right) \right]$$

$t - \lambda < 0$, $e^{(-ve)N} \longrightarrow 0$

$$\therefore M(t) = \left[\frac{\lambda}{t-\lambda} (0 - 1) \right] = \frac{\lambda}{\lambda - t}$$

$X \sim Exp(\lambda)$

$$M_X(t) = \begin{cases} \dfrac{\lambda}{\lambda - t} & t < \lambda \\ \infty & t \geq \lambda \end{cases}$$

IMP: $M_X(t)$ IS NOT ALWAYS FINITE!

??

$\boxed{\text{MOMENT GENERATING}}$ FUNCTION

RECALL : k^{th} MOMENT

$$\mathbb{E}(X^k)$$

REASON : IF $M_X(t)$ IS WELL-BEHAVED, THEN

$$M'(t) = \frac{d}{dt}\left[\mathbb{E}(e^{tX})\right] = \mathbb{E}\left[\frac{d}{dt}(e^{tX})\right] = \mathbb{E}\left[X e^{tX}\right]$$

TRUE IN IMP. CASES.

$$\therefore M'(0) = \mathbb{E}\left[X \cdot e^{0 \cdot X}\right] = \mathbb{E}[X] = \text{MEAN} \;/\; \text{1st MOMENT.}$$

MORE GENERALLY,

$$M^{(k)}(t) = \frac{d^k}{dt^k} \mathbb{E}\left[e^{tX}\right] \overset{?}{=} \mathbb{E}\left[\frac{d^k}{dt^k} e^{tX}\right] = \mathbb{E}\left[X^k e^{tX}\right]$$

$$\therefore \quad M^{(k)}(0) = \mathbb{E}\left[X^k e^{0 \cdot X}\right] = \mathbb{E}\left[X^k\right] \rightarrow k\text{th MOMENT.}$$

$$M(t) \rightsquigarrow M^{(k)}(0) = \mathbb{E}\left[X^k\right]$$

"MOMENT GEN. FUNCTION"

Fact 5.7. When the moment generating function $M(t)$ of a random variable X is finite in an interval around the origin, the moments of X are given by

$$E(X^n) = M^{(n)}(0).$$

NOTE: THIS ONLY WORKS IF ($\exists\, \epsilon > 0$)

$M(t) < \infty$ FOR $-\epsilon < t < \epsilon$

[IN OTHER WORDS: $M(t) < \infty$ IN AN INTERVAL AROUND $t = 0$.]

Example 5.8. (Moments of the Bernoulli distribution)

$$P(X=1) = p, \qquad P(X=0) = 1-p$$

$$M(t) = E(e^{tX}) = P(X=0)\cdot e^{t\cdot 0} + P(X=1)\, e^{t\cdot 1}$$

$$\therefore M(t) = (1-p) + pe^{t}$$

$$\therefore M^{(k)}(t) = pe^{t} \qquad (\text{IF } k \geq 1)$$

$$\therefore E[X^k] = M^{(k)}(0) = pe^{0} = p$$

OTOH : $E[X^k] = \cancel{P(X=0)\cdot 0^k} + P(X=1)\cdot 1^k = p$

ANSWERS MATCH.

Example 5.9. (Moments of the exponential distribution) Let $X \sim \mathrm{Exp}(\lambda)$. From Example 5.6 its m.g.f. is

$\lambda > 0$

$$M(t) = \begin{cases} \dfrac{\lambda}{\lambda - t}, & \text{if } t < \lambda \\ \infty, & \text{if } t \geq \lambda. \end{cases}$$

FINITE, IN $(-\infty, \lambda)$,

$\therefore$ ALSO IN $(-\epsilon, \epsilon)$ FOR $t < \lambda$

$$M'(t) = \frac{d}{dt}\left[\frac{\lambda}{\lambda - t}\right] = \frac{\lambda}{(\lambda - t)^2}$$

$$M''(t) = \frac{d}{dt}\left[\frac{\lambda}{(\lambda - t)^2}\right] = \frac{2\lambda}{(\lambda - t)^3}$$

$$M^{(3)}(t) = \frac{d}{dt}\left[\frac{2\lambda}{(\lambda-t)^3}\right]$$

$$= \frac{(1\cdot 2\cdot 3)\,\lambda}{(\lambda-t)^4} = \frac{6\lambda}{(\lambda-t)^4}$$

$$\vdots$$

$$M^{(k)}(t) = \frac{k!\,\lambda}{(\lambda-t)^{k+1}} \Rightarrow M^{(k)}(0) = \frac{k!\cdot\lambda}{\lambda^{k+1}} = \frac{k!}{\lambda^k}$$

$X \sim \mathrm{Exp}(\lambda)$

$$\Rightarrow \quad \mathbb{E}[X^k] = \frac{k!}{\lambda^k}$$

$\boxed{k=1}$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$\boxed{k=2}$

$$\mathbb{E}[X^2] = \frac{2}{\lambda^2}$$

$\mathrm{Var}(X) = 1/\lambda^2$

$$\mathbb{E}[X^2] = \mathrm{Var}(X) + (\mathbb{E}(X))^2$$
$$= 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$$

RECALL: (TAYLOR EXPANSION)

FOR "NICE" f,

$$f(t) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} t^n$$

$\therefore$ IF $M_X(t)$ IS "NICE",

$$M_X(t) = \sum_{n=0}^{\infty} \boxed{\frac{M^{(n)}(0)}{n!}} t^n = \underbrace{\sum_{n=0}^{\infty} \frac{\mathbb{E}[X^n]}{n!}}_{} t^n$$

Example 5.10. (Moments of the standard normal) Let $Z \sim \mathcal{N}(0, 1)$. From Example 5.5 we have $M_Z(t) = e^{t^2/2}$. Instead of differentiating this repeatedly, we can find the derivatives directly from the Taylor expansion.

$$M(t) = e^{t^2/2} = \sum_{k=0}^{\infty} \frac{(t^2/2)^k}{k!}$$

[TAYLOR FOR e^x]

$$= \sum_{k=0}^{\infty} \frac{t^{2k}}{2^k \cdot k!} = \sum_{n=0}^{\infty} \left[\begin{array}{ll} 0 & \text{IF } n \text{ ODD} \\ \dfrac{1}{2^{n/2} \cdot (n/2)!} & \text{IF } n \text{ EVEN} \end{array} \right] t^n = \sum_{n=0}^{\infty} \boxed{\frac{\mathbb{E}[X^n]}{n!}} t^n$$

$$\therefore \quad \mathbb{E}[X^n] = \begin{cases} 0 & \text{IF } n \text{ IS ODD} \\ \dfrac{n!}{2^{n/2} \cdot (n/2)!} = \dfrac{(2k)!}{2^k \cdot k!} & n = 2k \text{ IS EVEN} \end{cases}$$

MOMENTS OF STANDARD NORMAL.

$$\frac{\mathbb{E}[X^n]}{n!} = \frac{1}{2^{n/2} (n/2)!}$$

NOTE ① : $n = 2k+1$ IS ODD

$$E\left[X^{2k+1}\right] = \int_{-\infty}^{\infty} \boxed{x^{2k+1}\,\underbrace{\phi(x)}_{\text{p.d.f. OF } \mathcal{N}(0,1)}} \, dx = 0$$

ODD FUNCTION

NOTE ② :

$$n!! = \begin{cases} n(n-2)(n-4)\cdots 2 & \text{IF } n \text{ IS EVEN} \\ n(n-2)(n-4)\cdots 1 & \text{IF } n \text{ IS ODD} \end{cases}$$

$$\mathbb{E}[X^n] = \begin{cases} 0 & n \text{ IS ODD} \\ (n-1)!! & n \text{ IS EVEN} \end{cases}$$

MOMENTS OF STANDARD