MATH 201 (SUMMER 2023, SESH A2)

LECTURE 17 : 06/14/23

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

(1) OFFICE HOURS : TODAY: 11:15 AM – 12:15 PM (OR BY APPT.)

(2) UPCOMING DEADLINES : (i) WW 8 – TODAY (iii) WW 9 – SAT
(ii) HW 8 – THURS
NEXT WEEK'S DEADLINES ARE ALSO UP! [2HW, 1WW]

(3) CLASS ON T, JUNE 20th TO BE FLIPPED.

(4) PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 5.1 MOMENT GENERATING FUNCTION

(CONTD)

RECALL :

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]$$ (WHEN IT EXISTS)

THEN [IF $M_X(t) < \infty$ FOR ALL t AROUND 0],

$$\mathbb{E}[X^k] = M^{(k)}(0)$$ AND [IF M_X IS NICE]

$$M_X(t) = \sum_{k=0}^{\infty} M^{(k)}(0)\, t^k = \sum_{k=0}^{\infty} \mathbb{E}[X^k]\, t^k$$

* $X \sim Exp(\lambda)$

$$M_X(t) = \begin{cases} \dfrac{\lambda}{\lambda - t}, & t < \lambda \\ \infty, & t \geq \lambda \end{cases}$$

$$\mathbb{E}[X^k] = \frac{k!}{\lambda^k}$$

* $X \sim Pois(\lambda) \Rightarrow M_X(t) = exp[\lambda \cdot (e^t - 1)]$

$$= e^{\lambda[e^t - 1]}$$

$\{ exp[\cdot] = e^{(\cdot)} \}$

* $X \sim N(0,1) \Rightarrow M_X(t) = e^{t^2/2}$

$$E[X^n] = \begin{cases} 0 & \text{IF } n \text{ ODD} \\ (n-1)!! & \text{IF } n \text{ EVEN} \end{cases}$$

$(n-1)!! \to (n-1)(n-3)\cdots(1)$

RECALL :

$n!! = n(n-2)\cdots\cdot\left[{}^{2}/_{1}\right]$ n IS ODD/EVEN

EQUALITY IN DISTRIBUTION

Example 5.11. Define three random variables from three different experiments:

EQUALLY LIKELY

$$X = \begin{cases} 1, & \text{if a fair coin flip is heads} \\ 0, & \text{if a fair coin flip is tails,} \end{cases} \qquad Y = \begin{cases} 1, & \text{if a roll of a fair die is even} \\ 0, & \text{if a roll of a fair die is odd,} \end{cases}$$

$P(Y=0) = P(Y=1) = 1/2$

and
$$Z = \begin{cases} 1, & \text{if a card dealt from a deck is spades or clubs} \\ 0, & \text{if a card dealt from a deck is hearts or diamonds.} \end{cases}$$

$P(Z=0) = P(Z=1) = 1/2$

The random variables X, Y and Z have the same probability mass function $p(1) = p(0) = 1/2$. Thus any question about probabilities would have the same answer for each of the three random variables. ▲

$P(X=0) = P(X=1) = 1/2$

X, Y, Z ARE EQUAL IN DISTRIBUTION

Definition 5.12. Random variables X and Y are **equal in distribution** if $\underbrace{P(X \in B) = P(Y \in B)}_{\text{DISTRIBUTION OF } X \text{ \& } Y.}$ for all subsets B of $\mathbb{R}$. This is abbreviated by $X \overset{d}{=} Y$. ♣

Note in particular that the definition allows for the possibility that X and Y are defined on different sample spaces. In the context of Example 5.11 we can state that $X \overset{d}{=} Y \overset{d}{=} Z$.

WE HAVE IMPLICITLY USED THIS DEFN. ALREADY.

M.G.F.S UNIQUELY IDENTIFY DISTRIBUTIONS.

Fact 5.14. Let X and Y be two random variables with moment generating functions $M_X(t) = E(e^{tX})$ and $M_Y(t) = E(e^{tY})$. Suppose there exists $\delta > 0$ such that for $t \in (-\delta, \delta)$, $M_X(t) = M_Y(t)$ and these are finite numbers. Then X and Y are equal in distribution.

SPECIAL CASE (X HAS FINITE SUPPORT):

$$M_X(t) = E(e^{tX}) = \sum_k e^{kt} P(X = k).$$

Example 5.15. Suppose that X has moment generating function

$$M_X(t) = \frac{1}{5}e^{-17t} + \frac{1}{4} + \frac{11}{20}e^{2t}. \tag{5.5}$$

$$M_X'(t) = \frac{-17}{5}e^{-17t} + \frac{11}{10}e^{2t}$$

$$M'_X(0) = \frac{-17}{5} + \frac{11}{10} = \frac{-23}{10} = \mathbb{E}[X]$$

$$M_X''(t) = \frac{289}{5}e^{-17t} + \frac{11}{5}e^{2t} \Rightarrow \mathbb{E}[X^2] = M_X''(0) = \frac{289}{5} + \frac{11}{5} = 60$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 60 - \left(\frac{-23}{10}\right)^2$$

$\frac{1}{4} e^{0t}$

Example 5.15. Suppose that X has moment generating function

$$M_X(t) = \frac{1}{5}e^{-17t} + \frac{1}{4} + \frac{11}{20}e^{2t}. \tag{5.5}$$

$$M_X(t) = \mathbb{E}\left(e^{tX}\right) = \sum_k e^{tk} P(X = k)$$

SUPPORT OF $X \in \{-17, 0, 2\}$

$$P(X=-17) = \frac{1}{5}, \quad P(X=0) = \frac{1}{4}, \quad P(X=2) = \frac{11}{20}$$

REMARK :

$$M_X(0) = \mathbb{E}[e^{0 \cdot X}] = \mathbb{E}[e^0] = 1$$

TRUE NO MATTER WHAT DIST. OF X IS.

§ 8.3 SUMS & M.G.F.s

Fact 8.18. Suppose that X and Y are independent random variables with moment generating functions $M_X(t)$ and $M_Y(t)$. Then for all real numbers t,

$$M_{X+Y}(t) = M_X(t)M_Y(t). \tag{8.16}$$

$$e^{t\cdot(x+y)} = e^{tx+ty} = e^{tx}\cdot e^{ty}$$

Pf:

$$M_{x+y}(t) = \mathbb{E}\left[e^{t\cdot(X+Y)}\right] = \mathbb{E}\left[\boxed{e^{tX}}\cdot\boxed{e^{tY}}\right] = \underbrace{\mathbb{E}\left[e^{tX}\right]}_{M_X(t)}\cdot\underbrace{\mathbb{E}\left[e^{tY}\right]}_{M_Y(t)}$$

X & Y ARE INDEPENDENT.

$\boxtimes$

RECIPE : TO UNDERSTAND $X+Y$ (X, Y IND.) & CONVOLUTIONS, FIRST FIND M_X & M_Y & THEN LOOK FOR Z SATISFYING

$$M_Z(t) = M_X(t)\, M_Y(t)$$

$\Rightarrow$ BY FACT THAT m.g.f. DETERMINES DIST.

$$Z \stackrel{d}{=} X+Y$$

Example 8.19. (Convolution of Poisson random variables revisited) Suppose that $X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poisson}(\mu)$ and these are independent.

RECALL: $\boxed{X + Y \sim \text{Pois}(\lambda + \mu)}$

$$M_X(t) = \exp[\lambda(e^t - 1)]$$

$$M_Y(t) = \exp(\mu(e^t - 1)]$$

$$M_X(t)M_Y(t) = \exp[\lambda(e^t-1)] \cdot \exp[\mu(e^t-1)]$$

$$= \exp[\lambda(e^t-1) + \mu(e^t-1)]$$

$$= \exp[\underbrace{(\lambda+\mu)}(e^t-1)]$$

$M_Z(t)$, $Z \sim \text{Pois}(\lambda+\mu)$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = M_Z(t), \quad Z \sim Pois(\lambda + \mu)$$

$$\Rightarrow \quad X + Y \overset{d}{=} Z$$

$$\therefore \; X + Y \sim Pois(\lambda + \mu)$$

m.g.f. of $N(\mu, \sigma^2)$

$X \sim N(\mu, \sigma^2)$ & $Z \sim N(0,1)$

$X \overset{d}{=} \sigma Z + \mu$

$\left[\hat{X} = \frac{X-\mu}{\sigma} \sim N(0,1) \quad \therefore \frac{X-\mu}{\sigma} \overset{d}{=} Z \right]$

$M_Z(t) = e^{t^2/2}$

$$M_X(t) = \mathbb{E}\left[e^{tX}\right]$$

$$= \mathbb{E}\left[e^{t(\sigma Z + \mu)}\right] \quad \left(\because X \overset{d}{=} \sigma Z + \mu\right)$$

$$= \mathbb{E}\left[e^{(\sigma t)Z} \cdot \boxed{e^{t\mu}}\right] = e^{t\mu} \cdot \mathbb{E}\left[e^{(\sigma t)Z}\right]$$

DOES NOT DEPEND ON Z.

$\mathbb{E}\left[e^{tZ}\right]$

$t \mapsto \sigma t$

$$= e^{t\mu} M_Z(\sigma t) = e^{t\mu} \cdot e^{(\sigma t)^2/2}$$

$$= e^{\mu t + \sigma^2 t^2/2} = \exp\left[\mu t + \frac{\sigma^2 t^2}{2}\right]$$

Example 8.20. (Convolution of normal random variables revisited) Suppose that $X \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y \sim \mathcal{N}(\mu_2, \sigma_2^2)$, and these are independent.

RECALL :

$$X + Y \sim \mathcal{N}(\mu_1 + \mu_2 , \sigma_1^2 + \sigma_2^2)$$

[NO PROOF]

BY PREV. DISCUSSION,

$$M_X(t) = \exp\left[\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right] \qquad \left[e^{t \cdot \text{MEAN} + \frac{t^2 \cdot \text{VAR}}{2}}\right]$$

$$M_Y(t) = \exp\left[\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right]$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t)$$

$$= \exp\left[\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right] \cdot \exp\left[\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right]$$

$$= \exp\left[\left(\mu_1 t + \frac{\sigma_1^2 t^2}{2}\right) + \left(\mu_2 t + \frac{\sigma_2^2 t^2}{2}\right)\right]$$

$$= \exp\left[\underbrace{(\mu_1 + \mu_2)}_{\mu} t + \underbrace{(\sigma_1^2 + \sigma_2^2)}_{\sigma^2} \frac{t^2}{2}\right]$$

$$= M_Z(t) \qquad \left[Z \sim N(\mu, \sigma^2) = N\left(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2\right) \right]$$

$$\Rightarrow X + Y \overset{d}{=} Z \Rightarrow X + Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

Example 8.21. Let X and Y be independent with probability mass functions

$$p_X(1) = \tfrac{1}{3}, \quad p_X(2) = \tfrac{1}{4}, \quad p_X(3) = \tfrac{1}{6}, \quad p_X(4) = \tfrac{1}{4}$$

and

$$p_Y(1) = \tfrac{1}{2}, \quad p_Y(2) = \tfrac{1}{3}, \quad p_Y(3) = \tfrac{1}{6}.$$

Find the probability mass function of $X + Y$.

2 WAYS : (1) USE CONVOLUTIONS.

$X+Y \in \{2, 3, \dots, 7\}$

$$p_{X+Y}(n) = \sum_{k+l=n} p_X(k)\, p_Y(l)$$

e.g. $p_{X+Y}(2) = \sum_{k+l=2} p_X(k)\, p_Y(l) = p_X(1)\, p_Y(1) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$

$$P_{X+Y}(4) = \sum_{k+l=2} \left[\qquad \right]$$

$$(k,l) = (1,3), \quad (k,l) = (2,2), \quad (k,l) = (3,1)$$

$\therefore$

$$\begin{aligned}
P_{X+Y}(4) &= P_X(1)\,P_Y(3) + P_X(2)\,P_Y(2) + P_X(3)\,P_Y(1) \\
&= \frac{1}{3}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{3} + \frac{1}{6}\cdot\frac{1}{2} \\
&= \frac{2+3+3}{36} = \frac{8}{36} = \frac{2}{9}.
\end{aligned}$$

② USE m.g.f.

Example 8.21. Let X and Y be independent with probability mass functions

$$p_X(1) = \tfrac{1}{3}, \quad p_X(2) = \tfrac{1}{4}, \quad p_X(3) = \tfrac{1}{6}, \quad p_X(4) = \tfrac{1}{4}$$

and

$$p_Y(1) = \tfrac{1}{2}, \quad p_Y(2) = \tfrac{1}{3}, \quad p_Y(3) = \tfrac{1}{6}.$$

Find the probability mass function of $X + Y$.

$$M_X(t) = \sum_{k=1}^{4} e^{kt} \cdot p_X(k) = \frac{1}{3}e^{t} + \frac{1}{4}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{4}e^{4t}$$

lly

$$M_Y(t) = \frac{1}{2}e^{t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = \left(\frac{1}{3}e^{t} + \frac{1}{4}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{4}e^{4t}\right)\cdot\left(\frac{1}{2}e^{t} + \frac{1}{3}e^{2t} + \frac{1}{6}e^{3t}\right)$$

$$= \frac{1}{6}e^{2t} + \frac{1}{9}e^{3t} + \frac{1}{18}e^{4t} + \frac{1}{8}e^{3t} + \frac{1}{12}e^{4t}$$

$$+ \frac{1}{24}e^{5t} + \frac{1}{12}e^{4t} + \frac{1}{18}e^{5t} + \frac{1}{36}e^{6t}$$

$$+ \frac{1}{8}e^{5t} + \frac{1}{12}e^{6t} + \frac{1}{24}e^{7t}$$

$$= \frac{1}{6}e^{2t} + \frac{17}{72}e^{3t} + \frac{2}{9}e^{4t} + \frac{2}{9}e^{5t} + \frac{1}{9}e^{6t} + \frac{1}{24}e^{7t}$$

$$M_{X+Y}(t) = \frac{1}{6}e^{2t} + \frac{17}{72}e^{3t} + \frac{2}{9}e^{4t} + \frac{2}{9}e^{5t} + \frac{1}{9}e^{6t} + \frac{1}{24}e^{7t}$$

$$X+Y \in \{2, 3, 4, 5, 6, 7\}$$

$$P(X+Y=2) = \frac{1}{6} \qquad P(X+Y=3) = \frac{17}{72}$$

$$P(X+Y=4) = P(X+Y=5) = \frac{2}{9}$$

$$P(X+Y=6) = \frac{1}{9} \qquad P(X+Y=7) = \frac{1}{24}$$

MORE GENERALLY,

$$S = X_1 + X_2 + \cdots + X_n \qquad (X_j \text{ IND.})$$

$$\Rightarrow M_S(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t)$$

Example 8.22. Let $X_1, X_2, \ldots, X_{20}$ be independent random variables with probability mass function $P(X_i = 2) = \frac{1}{3}$ and $P(X_i = 5) = \frac{2}{3}$. Let $S = X_1 + \cdots + X_{20}$. Find the moment generating function of S.

X_j : $P(X_j = 2) = 1/3$, $P(X_j = 5) = 2/3$

$$M_{X_j}(t) = E\left(e^{tX_j}\right) = \frac{1}{3}e^{2t} + \frac{2}{3}e^{5t}$$

$$M_S(t) \prod_{j=1}^{20} M_{X_j}(t) = \prod_{j=1}^{20}\left(1/3\, e^{2t} + 2/3\, e^{5t}\right) = \left(\frac{1}{3}e^{2t} + \frac{2}{3}e^{5t}\right)^{20}$$

BREAK TILL

10:15

§ 9.1 ESTIMATING TAIL PROBABILITIES

RECALL: (RIGHT) TAIL OF X = $P(X \geq t)$

IN APPLICATIONS, IT IS OFTEN ENOUGH TO UPPER BOUND $P(X \geq t)$

SUCH ESTIMATES ARE CALLED TAIL INEQUALITIES.

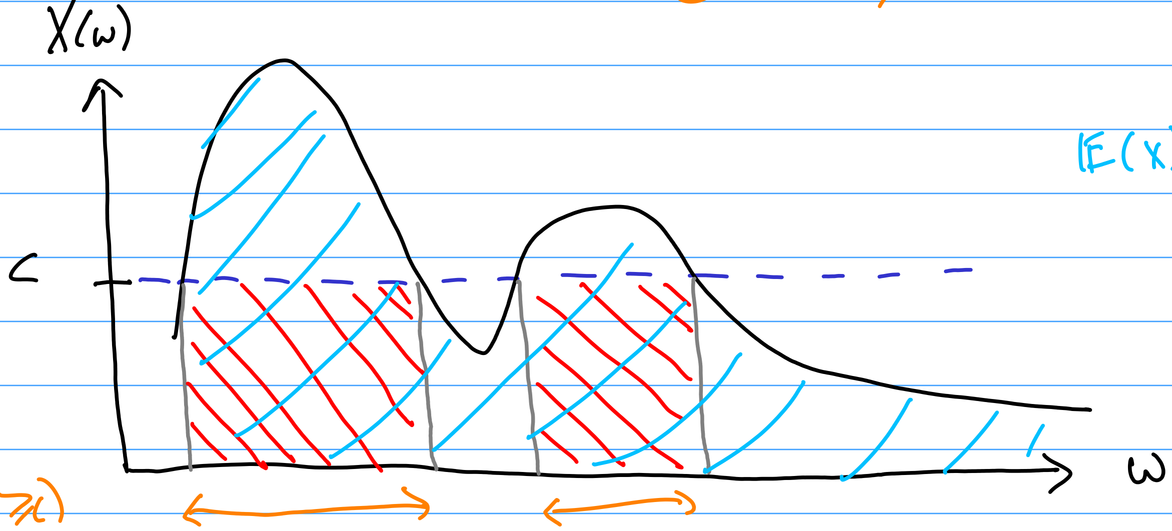
BASIC : MARKOV

Theorem 9.2. (Markov's inequality) *Let X be a nonnegative random variable. Then for any $c > 0$*

$$P(X \geq c) \leq \frac{E[X]}{c}. \tag{9.1}$$

[BOUND (OBV DEPENDS ON X, c]

IDEA OF PF :



$\mathbb{E}(X)$ = AREA OF

$c \cdot P(X \geq c)$ = AREA OF

PICTURE TELLS YOU:

$$c \cdot P(X \geq c) \leq \mathbb{E}(X)$$

$$\Rightarrow P(X \geq c) \leq \frac{\mathbb{E}(X)}{c}$$

Pf FOR X CONT.:

$B = \{ x \in \mathbb{R} : x \geq c \}$

$X \geq 0 \Rightarrow f(x) = 0$ IF $x < 0$

$$c \cdot P(X \geq c) = c\, P(X \in B) = c \int_B f(x)\, dx$$

$cf(x) \leq x \cdot f(x)$ FOR $x \in B$

$$= \int_c^\infty \boxed{c f(x)}\, dx \leq \int_c^\infty x f(x)\, dx \leq \int_0^\infty x f(x)\, dx = \mathbb{E}[X]$$

∎

EFFECTIVENESS OF MARKOV VARIES.

e.g. $X \sim \text{Ber}(p)$ $\quad\quad$ $\mathbb{E}[X] = 0 \cdot (1-p) + 1 \cdot p = p$

$P(X \geq 1)$

→ MARKOV: $P(X \geq 1) \leq \dfrac{\mathbb{E}[X]}{1} = p. \therefore P(X \geq 1) \leq p$

SHARP!

→ EXACT: $P(X \geq 1) = P(X = 1) = p$

VERY EFFECTIVE.

$P(X \geq 0.01)$

MARKOV:

$$P(X \geq 0.01) \leq \frac{E(X)}{0.01} = \overbrace{100p}^{\text{BIGGER THAN 1, UNLESS } p < 0.01}$$

FAR FROM SHARP

EXACT:

$$P(X \geq 0.01) = P(X = 1) = p$$

NOT VERY EFFECTIVE.

Example 9.4. An ice cream parlor has, on average, 1000 customers per day. What can be said about the probability that it will have at least 1400 customers tomorrow.

$X \rightarrow$ # OF CUSTOMERS TOMORROW

$\mathbb{E}[X] = 1000$

$$P(X \geq 1400) \leq \frac{\mathbb{E}[X]}{1400} = \frac{1000}{1400} = \frac{5}{7}$$

(MARKOV)

$\therefore P(X \geq 1400) = 5/7.$

(NOTE: $|X-\mu| \geq c$

$\Rightarrow$ EITHER $X \geq \mu + c$

OR $X \leq \mu - c$)

MORE REFINED: CHEBYSHEV

Theorem 9.5. (Chebyshev's inequality) *Let X be a random variable with a finite mean μ and a finite variance σ^2. Then for any $c > 0$ we have*

$$P(|X-\mu| \geq c) \leq \frac{\sigma^2}{c^2}. \tag{9.2}$$

COROLLARY: $P(X \geq \mu + c),\ P(X \leq \mu - c) \leq \sigma^2 / c^2$

$\{|X-\mu| \geq c\} = \{X \geq \mu + c\} \cup \{X \leq \mu - c\}$

RECALL: VARIANCE MEASURES SPREAD

Pf : FOLLOWS FROM MARKOV ON $Y = (X-\mu)^2$

$$Y = (X-\mu)^2 \geqslant 0 \Rightarrow \mathbb{E}(Y) = \mathbb{E}[(X-\mu)^2] = \mathrm{Var}(X) = \sigma^2$$

$\therefore$ MARKOV APPLIES.

$$P(|X-\mu| \geqslant c) = P((X-\mu)^2 \geqslant c^2)$$

$$= P(Y \geqslant c^2)$$

$$\underset{\text{(MARKOV)}}{\leqslant} \frac{\mathbb{E}[Y]}{c^2} = \frac{\sigma^2}{c^2}$$

$$\therefore P(|X-\mu| \geqslant c) \leqslant \sigma^2 / c^2$$

Example 9.6. Suppose a nonnegative random variable X has mean 50. $\mathbb{E}(X) = 50$

(a) Give an upper bound for the probability $P(X \geq 60)$.

(b) Suppose that we also know $\mathrm{Var}(X) = 25$. How does your bound for $P(X \geq 60)$ change?

(c) Suppose that we also know that X is $\boxed{\text{binomially distributed.}}$ Compare the upper bounds from (a) and (b) with the normal approximation and the precise value of $P(X \geq 60)$.

(a) : MARKOV — $P(X \geqslant 60) \leqslant \dfrac{\mathbb{E}(X)}{60} = \dfrac{50}{60} = \dfrac{5}{6}$

(b) $\mathrm{Var}(X) = 25 \longrightarrow$ CHEBYSHEV! $X \geqslant 60 = \underbrace{50}_{\mu} + \underbrace{10}_{c}$

$$P(X \geqslant 60) = P(X \geqslant 50 + 10) \leqslant P(|X - \underbrace{50}_{\mu}| \geqslant \underbrace{10}_{c}) \leqslant \frac{\sigma^2}{c^2} = \frac{25}{10^2} = \frac{1}{4}$$

(c) NORMAL :

$$P(X \geq 60)$$

$$\hat{X} = \frac{X-\mu}{\sigma} = \frac{X-50}{\sqrt{25}} = \frac{X-50}{5}$$

$$X \geq 60 \iff \frac{X-50}{5} \geq \frac{60-50}{5} = 2$$

$$P(X \geq 60) = P(\hat{X} \geq 2) \simeq P(Z \geq 2) = P(2 \leq Z < \infty)$$

$[Z \sim N(0,1)]$

$$= \Phi(\infty) - \Phi(2)$$

$$= 1 - \Phi(2) \approx 0.0228$$

CONTINUITY CORRECTION

$$P(X \geq 60) = P(X \geq 59.5)$$

$$= P\left(\hat{X} \geq \frac{59.5 - 50}{5}\right)$$

$$= P(\hat{X} \geq 1.9) \approx P(Z \geq 1.9)$$

$$= 1 - \Phi(1.9)$$

$$\approx 0.0287$$

EXACT : $X \sim \mathrm{Bin}(n, p)$

$$np = \mathbb{E}(X) = 50$$

$$np(1-p) = \mathrm{Var}(X) = 25$$

$$\Rightarrow \quad 50(1-p) = \underbrace{np}_{50}(1-p) = 25$$

$$\Rightarrow \quad p = \frac{1}{2}, \quad n = \frac{50}{p} = \frac{50}{1/2} = 100$$

$$X \sim \mathrm{Bin}(100, 1/2) \Rightarrow P(X \geq 60) = \sum_{k=60}^{100} \binom{100}{k} 2^{-100} = 0.0284$$

$\mathbb{E}(X)$	MARKOV	$\leq 5/6 \approx 0.8333$
$\mathbb{E}(X)$, $Var(X)$	CHEBYSHEV	$\leq 1/4 = 0.25$
USES APPROX. $Bin(n,p)$	NORMAL	≈ 0.0228
	C.C. NORMAL	≈ 0.0287
ALL INFO BEING USED	EXACT	≈ 0.0284

Example 9.7. Continuing Example 9.4, we now suppose that the variance of the number of customers to the ice cream parlor on a given day is 200. What can be said about the probability that there will be between 950 and 1050 customers tomorrow? What can we say about the probability that at least 1400 customers arrive?

$X \to$ # OF CUSTOMERS TOMORROW

$\mathbb{E}[X] = 1000 = \mu$

$\mathrm{Var}[X] = 200 = \sigma^2$

$P(X \geq 1400) = P(X - 1000 \geq 400)$

$\leq P(|X - \underbrace{1000}_{\mu}| \geq \underbrace{400}_{c}) \leq \frac{\overset{\sigma^2}{200}}{\underbrace{400^2}_{c^2}} = \frac{1}{800}$

COMPARE: $\boxed{5/7}$

$$\geq \frac{23}{25} = 0.92$$

$$P(950 < X < 1050)$$

$$= P(950 - 1000 < X - 1000 < 1050 - 1000)$$

$$= P(-50 < X - \mu < 50)$$

$$= P(|X - \mu| < 50)$$

$$= 1 - \underbrace{P(|X - \mu| \geq 50)}_{\text{CHEBYSHEV}}$$

$$\geq 1 - \frac{\overbrace{200}^{\sigma^2}}{\underbrace{50^2}_{c^2}} = 1 - \frac{2}{25} = \frac{23}{25}$$

Example 9.8. Suppose the random variable X is discrete and $\boxed{\text{symmetric}}$, meaning that $P(X = a) = P(X = -a)$ for any a. Estimate $P(X \geq 5)$ if $\mathrm{Var}(X) = 3$.

$\sigma^2 = 3$

$$\mathbb{E}(X) = \sum_k k\, P(X = k)$$

$$= \sum_{k>0} k\, P(X = k) + \sum_{k<0} k \cdot P(X = k)$$

$$= \sum_{k>0} k\, P(X = k) + \sum_{k>0} (-k)\, P(X = -k)$$

$$= \sum_{k>0} P(X = k)\,[k + (-k)] = 0$$

$$P(X \geq 5) \quad ?$$

$$P(X \geq \underset{\mu}{0} + \underset{c}{5}) \leq \frac{\overset{\sigma^2}{3}}{\underset{c^2}{5^2}} = \frac{3}{25} = 0.12$$

$$P(|X| \geq 5) = P(X \geq 5) + \underbrace{P(X \leq -5)}_{= P(X \geq 5)}$$

$$= 2P(X \geq 5)$$

$$P(X \geq 5) = \frac{1}{2} P(|X| \geq 5) = \frac{1}{2} P(|X - 0| \geq 5) \overset{\text{CHEB.}}{\leq} \frac{1}{2} \cdot \frac{3}{5^2} = 0.06$$

e.g.: $Y \sim \text{Geom}(1/6)$, $P(Y \geq 7)$

(a) MARKOV

$$\mu = \mathbb{E}[Y] = 1/p = 1/(1/6) = 6$$

$$Y \in \{1, 2, 3, \ldots\} \geq 0$$

$$P(Y \geq 7) \leq \frac{\mathbb{E}[Y]}{7} = \frac{6}{7}$$

$$\sigma^2 = \text{Var}(X) = \frac{1-p}{p^2} = \frac{5/6}{(1/6)^2} = 30$$

(b) CHEBYSHEV

$$Y \geq 7 \Rightarrow Y - 6 \geq 1 \Rightarrow |Y - 6| \geq 1$$

$$P(Y \geq 7) \leq P(|Y - \underset{\mu}{6}| \geq 1) \leq \frac{\overset{\sigma^2}{30}}{\underset{c^2}{1^2}} = 30$$

$$P(Y \geq 7) = P(\text{FFFFFF})$$

$$= \left(\frac{5}{6}\right)^6$$

TRY THIS.

$$P(Y \geq 60)$$