MATH 201 (SUMMER 2023, SESH A2)

LECTURE 18 : 06/15/23

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. UPCOMING DEADLINES:
   - (i) HW 8 – TODAY
   - (ii) WW 9 – SAT
   - (iii) WW 10 – TUES
   - (iv) HW 9, HW 10* – WED

   * : EXTRA CREDIT

2. CLASS ON T, JUNE 20th MAY BE FLIPPED.

3. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 9.2 LAW OF LARGE NUMBERS (GENERAL CASE)

RECALL: IF $X_1, X_2, \ldots \sim \boxed{Ber(p)}$, i.i.d. → INDEPENDENT AND IDENTICALLY DISTRIBUTION

WHAT IF REPLACED BY OTHER DIST.?

[$S_n \sim Bin(n,p)$] $S_n = X_1 + X_2 + \cdots + X_n$.

THEN FOR ANY $\epsilon > 0$, FIXED

$$\lim_{n\to\infty} P\left(\left| \underbrace{\frac{S_n}{n}}_{\text{OBSERVED FRACTION OF SUCCESS.}} - \overbrace{p}^{\text{THEORETICAL FRACTION OF SUCCESS.}} \right| < \epsilon \right) = 1$$

[WEAK L.L.N. FOR BINOMIAL DISTRIBUTION]

NOW,

$$S_n = X_1 + X_2 + \cdots + X_n$$

X_j i.i.d.,

$\mu = \mathbb{E}[X_j] < \infty$

$\sigma^2 = \mathrm{Var}[X_j] < \infty$

OBSERVED AVERAGE $\rightarrow$ $\bar{X}_n = \dfrac{S_n}{n}$

THEORETICAL AVG. $\rightarrow$ $\mathbb{E}[X_1] = \mathbb{E}[\bar{X}_n] = \mu$

$$\boxed{\bar{X}_n \longrightarrow \mu \quad (\text{w/ PROB } 1 \text{ AS } n \longrightarrow \infty)}$$

Theorem 9.9. (Law of large numbers with finite variance) *Suppose that we have i.i.d. random variables $X_1, X_2, X_3, \ldots$ with finite mean $E[X_1] = \mu$ and finite variance* $\mathrm{Var}(X_1) = \sigma^2$. *Let $S_n = X_1 + \cdots + X_n$. Then for any fixed $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} P\left(\left|\underbrace{\boxed{\frac{S_n}{n}}}_{\bar{X}_n} - \mu\right| < \varepsilon\right) = 1. \tag{9.4}$$

Pf.

NEED TO BOUND

$$P\left(\left|\underbrace{\frac{S_n}{n}}_{\bar{X}_n} - \mu\right| \geq \epsilon\right)$$

(i.e. SMALL)

[$\because$ $\left|\frac{S_n}{n} - \mu\right| \geq \epsilon$ & $\left|\frac{S_n}{n} - \mu\right| < \epsilon$ PARTITION Ω]

$$\bar{X}_n = \frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

L.O.E. $\Rightarrow$ $$\mathbb{E}[\bar{X}_n] = \mathbb{E}\left[\frac{X_1 + \cdots + X_n}{n}\right] = \frac{\mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]}{n}$$

$$= \frac{\overbrace{\mu + \cdots + \mu}^{n}}{n} = \frac{n\mu}{n} = \mu.$$

$$P\left(\left|\bar{X}_n - \mathbb{E}\bar{X}_n\right| \geq \epsilon\right)$$ CHEBYSHEV!

$$\mathrm{Var}(\overline{X}_n) = \mathrm{Var}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right]$$

$$= \frac{1}{n^2}\mathrm{Var}[X_1 + \cdots + X_n] \qquad \left(\because \mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)\right)$$

$$= \frac{1}{n^2}\left(\mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n]\right) \qquad [X_j \text{ ARE INDEP.}]$$

$$= \frac{1}{n^2}\left(\underbrace{\sigma^2 + \cdots + \sigma^2}_{n}\right)$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

CHEBYSHEV:

$$P\left(|X - \mathbb{E}X| \geq c\right) \leq \frac{Var(X)}{c^2}$$

$$\underset{c \to \epsilon}{X \to \overline{X}_n}, \quad \mathbb{E}[X] = \mu, \quad Var(X) = \frac{\sigma^2}{n},$$

$$P\left(|\overline{X}_n - \mu| \geq \epsilon\right) \leq \frac{\sigma^2/n}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| < \epsilon\right) = 1 - P\left(|\overline{X}_n - \mu| \geq \epsilon\right) \geq 1 - \frac{\sigma^2}{n\epsilon^2}$$

$$1 \geq P\Big(\frac{S_n}{n} - \mu \Big| < \epsilon \Big) \geq 1 - \underbrace{\frac{\sigma^2}{n\epsilon^2}}_{\to 0 \ (\text{AS } n \to \infty)}$$

AS $n \to \infty$, BOTH END-POINTS $\to 1$

$$\therefore \lim_{n \to \infty} P\left(\left| \frac{S_n}{n} - \mu \right| < \epsilon \right) = 1$$

[SQUEEZE THEOREM].

$\square$

HEIGHT.

Example 9.10. Suppose we want to estimate the mean of a random variable $\boxed{X}$ from a finite number of independent samples from a population, using the sample mean. For example, perhaps we are measuring average ~~IQ score~~, average income, etc. Suppose also that we know an upper bound on the variance of X: $\mathrm{Var}(X) \leq \hat{\sigma}^2$. (This happens for example when $|X|$ is bounded by $\hat{\sigma}$.) Show that for a large enough sample the sample mean is within $\boxed{0.05}$ ϵ of the correct value with a probability larger than 0.99. How many samples do we need to take to be at least 99% sure that our estimated value is within 0.05 of the correct value?

$$|X| \leq \hat{\sigma}, \quad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\leq \mathbb{E}(X^2) \leq \mathbb{E}(\hat{\sigma}^2) = \hat{\sigma}^2$$

$X_1, X_2, \ldots$

$$\bar{X}_n = \frac{S_n}{n} = \frac{X_1 + \cdots + X_n}{n}. \qquad \epsilon = 0.05$$

$$\mu = \mathbb{E}[X_i]$$

$$\lim_{n \to \infty} P\left(|\bar{X}_n - \mu| < 0.05\right) = 1$$ [BY L.L.N.]

Q: HOW LARGE DOES n HAVE TO BE TO GUARANTEE

$$\underbrace{P\left(|\bar{X}_n - \mu| < 0.05\right) \geq 0.99}$$

LOWER BOUND $\iff$ UPPER BOUNDING

CHEBYSHEV. $P\left(|X_n - \mu| \geq 0.05\right)$

i.e., WE WANT

$$P\left(\left|\bar{X}_n - \mu\right| \geq 0.05\right) < 0.01$$

CHEBYSHEV:

$$P\left(\left|\bar{X}_n - \mu\right| \geq 0.05\right) \leq \frac{\text{Var}(\bar{X}_n)}{(0.05)^2}$$

$\mu = \mathbb{E}[X_1] = \mathbb{E}[\bar{X}_n]$

$$\text{Var}(\bar{X}_n) = \frac{\boxed{\sigma^2}}{n} \qquad \text{Var}(X_1) \qquad = \frac{\text{Var}[X_1]}{n} \leq \frac{\hat{\sigma}^2}{n}$$

$$P\left(|\bar{X}_n - \mu| \geq 0.05\right) \leq \frac{\mathrm{Var}(\bar{X}_n)}{(0.05)^2} \leq \frac{\hat{\sigma}^2}{n(0.05)^2}$$

WANT TO
MAKE
RHS < 0.01

$$\frac{\hat{\sigma}^2}{n(0.05)^2} < 0.01$$

$$\Rightarrow \quad n > \frac{\hat{\sigma}^2}{(0.01)(0.05)^2} = 40000\,\hat{\sigma}^2$$

e.g. FOR HEIGHT, $\hat{\sigma}^2 \leq 100$

($\because$ MOST PEOPLE ARE $\leq$ 10 ft)

$$n \geq (40000)(100) = 4,000,000$$

BREAK TILL
9:45 AM

§ 9.3 CENTRAL LIMIT THEOREM

WE NOW GENERALIZE C.L.T. TO ANY FINITE VARIANCE RANDOM VARIABLE (RECALL: WE DISCUSSED IT FOR Bin(n,p))

Theorem 9.11. (Central limit theorem) *Suppose that we have i.i.d. random variables $X_1, X_2, X_3, \ldots$ with finite mean $E[X_1] = \mu$ and finite variance $\mathrm{Var}(X_1) = \sigma^2$. Let $S_n = X_1 + \cdots + X_n$. Then for any fixed $-\infty \le a \le b \le \infty$ we have*

$$\lim_{n\to\infty} P\left(a \le \boxed{\frac{S_n - n\mu}{\sigma\sqrt{n}}} \le b\right) = \Phi(b) - \Phi(a) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy. \tag{9.6}$$

$$\frac{S_n - \mathbb{E}S_n}{\sqrt{\mathrm{Var}\, S_n}} = \hat{S}_n$$

$$\hat{S}_n \simeq N(0,1)$$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n\mu$$

$$\mathrm{Var}[S_n] = \mathrm{Var}[X_1 + \cdots + X_n] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n] = n\sigma^2$$

Example 9.12. We roll 1000 dice and add up the values. Estimate the probability that the sum is at least 3600.

$X_j \to j^{th}$ DIE ROLL.

$X_j \in \{1, 2, \ldots, 6\}$ (UNIFORM)

$X_j \to$ INDEP.

$$\mathbb{E}(X_1) = 1 \cdot P(X_1 = 1) + 2 \cdot P(X_1 = 2) + \cdots + 6 \cdot P(X_1 = 6)$$

$$= 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \cdots + 6 \cdot \frac{1}{6} = \frac{1}{6}[1 + 2 + \cdots + 6]$$

$$= 7/2$$

$$\mathbb{E}(X_1^2) = 1^2 \cdot 1/6 + 2^2 \cdot 1/6 + \cdots + 6^2/6 \quad ; \quad \mathrm{Var}(X_1) = 35/12$$

Example 9.12. We roll 1000 dice and add up the values. Estimate the probability that the sum is at least 3600.

$$S = X_1 + X_2 + \cdots + X_{1000}$$

$$\mathbb{E}[X_1] = \underbrace{7/2}_{\mu}, \qquad \mathrm{Var}[X_1] = \underbrace{35/12}_{\sigma^2}$$

$$P(\underbrace{S \geq 3600}_{\text{STANDARDIZE}})$$

$$\mathbb{E}(S) = 1000\,\mathbb{E}[X_1] = 3500$$

$$\mathrm{Var}(S) = 1000\,\mathrm{Var}[X_1] = \frac{35000}{12}$$

$\hat{S}$.

$$\hat{S} = \frac{S - ES}{\sqrt{\operatorname{Var} S}} = \frac{S - 3500}{\sqrt{35000/12}}$$

$$P(S \geq 3600) = P\left[S - 3500 \geq 100\right]$$

$$= P\left\{\boxed{\overset{\hat{S}}{\frac{S-3500}{\sqrt{35000/12}}}} \geq \frac{100}{\sqrt{35000/12}}\right\}$$

$\hat{S} \simeq N(0,1)$

$$= P\left(\hat{S} \geq \frac{100}{\sqrt{35000/12}}\right) \approx P\left(Z \geq \frac{100}{\sqrt{35000/12}}\right)$$

$$P\left(Z \geqslant \frac{100}{\sqrt{35000/12}}\right) = \Phi(\infty) - \Phi\left(\frac{100}{\sqrt{35000/12}}\right)$$

$$= 1 - \Phi\left(\frac{100}{\sqrt{35000/12}}\right)$$

$$\simeq 1 - \Phi(1.852)$$

$$\simeq 0.03$$

$$\therefore P(S \geqslant 3600) \simeq 0.03$$

BOOK DOES IT IN HOURS

Example 9.13. A new diner specializing in waffles opens on our street. It will be open 24 hours a day, seven days a week. It is assumed that the inter-arrival times between customers will be i.i.d. exponential random variables with mean 10 minutes. Approximate the probability that the 120th customer will arrive after the first $\boxed{\text{21 hours}}$ of operation.

$21 \times 60 = 1260$ min.

$S_n \rightarrow$ TIME AT WHICH CUSTOMER n ARRIVES

$$S_n - S_{n-1} \sim \text{Exp}\left(\frac{1}{10}\right) \qquad \mathbb{E}\left[\text{Exp}(\lambda)\right] = 1/\lambda = 10$$

$$\left.\begin{aligned} S_n - S_{n-1} &= X_n \\ S_1 &= X_1 \end{aligned}\right\} \Rightarrow \begin{aligned} S_n &= X_n + S_{n-1} \\ &= X_n + X_{n-1} + S_{n-2} \\ &= X_n + X_{n-1} + \cdots + X_1 \end{aligned}$$

$$X_j = S_j - S_{j-1} \overset{i.i.d.}{\sim} \mathrm{Exp}(1/10)$$

CLT APPLIES TO S_n !

Q: $P(\underbrace{S_{120}}_{\text{TIME AT WHICH 120th CUSTOMER ARRIVES}} \geq \underbrace{1260}_{21 \text{ HRS}})$ $[S_{120} = X_1 + X_2 + \dots + X_{120}]$

$$\mathbb{E}(S_{120}) = 120\,\mathbb{E}[X_1] = 120 \cdot 10 = 1200$$

$$\mathrm{Var}(S_{120}) = 120\,\mathrm{Var}(X_1) \qquad [\because X_j \text{ ARE INDEP.}]$$

RECALL : $\mathrm{Var}\left[\mathrm{Exp}(\lambda)\right] = \dfrac{1}{\lambda^2}$

$$\therefore \mathrm{Var}[X_1] = \frac{1}{(1/10)^2} = 100$$

$$\therefore \mathrm{Var}(S_{120}) = 120 \cdot 100 = 12000$$

$$P(S_{120} \geq 1260) = P(S_{120} - 1200 \geq 60)$$

$$= P\left(\frac{S_{120} - 1200}{\sqrt{12000}} \geq \frac{60}{\sqrt{12000}}\right) \approx P\left(Z \geq \frac{60}{\sqrt{12000}}\right)$$

$\hat{S}_{120} \approx N(0,1)$

$$= 1 - \Phi\left(\frac{60}{\sqrt{12000}}\right)$$

$$\simeq 1 - \Phi(0.55)$$

$$\simeq 1 - 0.71 \simeq 0.29.$$

ACTUAL PROBABILITY : 0.285

RECALL :

① $\text{Bin}(n,p) \approx \text{Pois}(np)$ IF $np^2 \ll 1$

② $\text{Bin}(n,p) \approx \mathcal{N}(np, npq)$ IF $npq \gg 1$

$[q := 1-p]$

Q: CAN BOTH HAPPEN SIMULTANEOUSLY?

A: YES! e.g. $p = \lambda/n$ $\quad np^2 = n \cdot \frac{\lambda^2}{n^2} = \boxed{\frac{\lambda^2}{n}} \to$ SMALL

& $\quad npq = n \cdot \frac{\lambda}{n}\left(1 - \frac{\lambda}{n}\right) = \boxed{\lambda\left(1 - \frac{\lambda}{n}\right)} \to$ LARGE

$p = \lambda/n \quad \Rightarrow \quad np = \lambda$

$$\begin{array}{c} \mathrm{Bin}(n,p) \approx \mathrm{Pois}(np) = \mathrm{Pois}(\lambda) \\ \wr\wr \\ N(np,\ npq) \approx N(\lambda,\ \_\_) \end{array}$$

Example 9.14. (Normal approximation of a Poisson)

RECALL: SUMS OF POISSON VARIABLES ARE POISSON (W/ PARAMS ADDED)

$X_1, X_2, \ldots \overset{iid}{\sim} \text{Pois}(1)$

$S_n = X_1 + X_2 + \cdots + X_n \sim \text{Pois}(\underbrace{1+1+\cdots+1}_{n})$

$\Rightarrow \quad S_n \sim \text{Pois}(n)$

$\therefore$ CLT. $\quad P\left(\dfrac{S_n - n\cdot 1}{\sqrt{n}\cdot 1} \leq a\right) \approx \Phi(a)$

IN GENERAL.

IF $Y \sim Pois(\lambda)$

$$P\left(\boxed{\frac{Y-\lambda}{\sqrt{\lambda}}} \le a\right) \xrightarrow{\lambda \to \infty} \Phi(a)$$

$$\hat{Y} = \frac{Y - \mathbb{E}[Y]}{\sqrt{\mathrm{Var}[Y]}}$$

WEAKER VERSION

Theorem 9.11. (Central limit theorem) *Suppose that we have i.i.d. random variables* $X_1, X_2, X_3, \dots$ ~~*with finite mean* $E[X_1] = \mu$ *and finite variance* $\mathrm{Var}(X_1) = \sigma^2$~~. *Let* $S_n = X_1 + \cdots + X_n$. *Then for any fixed* $-\infty \le a \le b \le \infty$ *we have*

$$\lim_{n\to\infty} P\left(a \le \frac{S_n - n\mu}{\sigma\sqrt{n}} \le b\right) = \Phi(b) - \Phi(a) = \int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy. \tag{9.6}$$

WITH $M_{X_1}(t) < \infty$
FOR
$-\epsilon \le t \le \epsilon$

FINITE
m.g.f.

RECALL :

m.g.f. DETERMINES DISTRIBUTION.

$$\therefore \; M_X(t) = M_Y(t) \implies X \stackrel{d}{=} Y.$$

Q : $M_X(t) \approx M_Y(t) \stackrel{?}{\implies} X \stackrel{d}{\approx} Y$? (STABILITY?)

A : YES! s.t. FOR e.g. $X_1, X_2, X_3, \ldots$

$$M_{X_j}(t) \longrightarrow M_X(t) \implies X_j \xrightarrow{d} X$$

Theorem 9.15. (Continuity theorem for moment generating functions) *Suppose the random variable X has a continuous cumulative distribution function, and that its moment generating function $M_X(t)$ is finite in an interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Assume further that the moment generating functions of the random variables $Y_1, Y_2, Y_3, \ldots$ satisfy*

$$\lim_{n\to\infty} M_{Y_n}(t) = M_X(t)$$

$(M_{Y_n} \longrightarrow M_X)$

for all t in the interval $(-\varepsilon, \varepsilon)$. Then for any $a \in \mathbb{R}$

$$\lim_{n\to\infty} P(Y_n \le a) = P(X \le a).$$

$(Y_n \xrightarrow{d} X)$

$\boxed{\text{BLACK BOX}}$

Theorem 9.16. *Assume that the moment generating functions of the random variables $Y_1, Y_2, Y_3, \ldots$ satisfy*

$M_{Y_n} \longrightarrow M_Z$

$$\lim_{n\to\infty} M_{Y_n}(t) = \boxed{e^{\frac{t^2}{2}}} \tag{9.7}$$

$\rightarrow M_Z(t)$, $Z \sim N(0,1)$

for all t in an interval $(-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Then for any $a \in \mathbb{R}$

$Y_n \xrightarrow{d} N(0,1)$

$$\lim_{n\to\infty} P(Y_n \le a) = \Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy.$$

$\therefore$ IT SUFFICES TO SHOW

$$M_{\hat{S}_n}(t) \longrightarrow e^{t^2/2} \quad \text{FOR ALL } -\epsilon \leq t \leq \epsilon.$$

($\because$ CONT. THM. $\Rightarrow$ $\hat{S}_n \xrightarrow{d} N(0,1)$)

PROOF SKETCH :

$(X_j \rightarrow \text{i.i.d})$

$$S_n = (X_1 + X_2 + \cdots + X_n)$$

$$\hat{S}_n = \frac{S_n - ES_n}{\sqrt{\operatorname{Var} S_n}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

[$\because$ INDEP.]

$$M_{S_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_n}(t)$$
$$= [M_{X_1}(t)]^n$$

RECALL :

$$M_{aY+b}(t) = \mathbb{E}\left[e^{t(aY+b)}\right]$$

$$= \mathbb{E}\left[e^{atY} \cdot e^{tb}\right]$$

$$= e^{tb} \cdot \mathbb{E}\left[e^{atY}\right]$$

$$= e^{tb} M_Y(at)$$

$$\left[\begin{array}{c} Y = S_n \\ b = -n\mu/\sigma\sqrt{n} \\ a = 1/\sigma\sqrt{n} \end{array}\right]$$

$$\therefore \; M_{\hat{S}_n}(t) = M_{\left[\frac{S_n - n\mu}{\sigma\sqrt{n}}\right]}(t) = e^{\left[-tn\mu/\sigma\sqrt{n}\right]} \cdot M_{S_n}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$M_{\hat{S}_n}(t) = e^{-n\mu t/\sigma\sqrt{n}} \left[M_{X_1}\left(\frac{t}{\sigma\sqrt{n}} \right) \right]^n$$

$$= e^{-n\mu t/\sigma\sqrt{n}} \left[\mathbb{E}\left[\exp\left[\frac{t X_1}{\sigma\sqrt{n}} \right] \right] \right]^n$$

$$= \mathbb{E}\left[\exp\left[\frac{t\,(X_1 - \mu)}{\sigma\sqrt{n}} \right] \right]^n$$

RECALL: $\exp(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots$

??

$$\exp\left[\frac{t\,(X_1-\mu)}{\sigma\sqrt{n}}\right] \approx 1 + \frac{t\,(X_1-\mu)}{\sigma\sqrt{n}} + \frac{t^2\,(X_1-\mu)^2}{2n\sigma^2}$$

$$\mathbb{E}\left[\exp\left[\frac{t\,(X_1-\mu)}{\sigma\sqrt{n}}\right]\right] \approx 1 + \cancel{\frac{t}{\sigma\sqrt{n}}\,\underbrace{\mathbb{E}(X_1-\mu)}} + \frac{t^2}{2n\cancel{\sigma^2}}\,\underbrace{\cancel{\mathbb{E}\left((X_1-\mu)^2\right)}}_{\sigma^2}$$

$\mathbb{E}X_1 - \mathbb{E}\mu = \mu - \mu = 0$

$$\approx 1 + \frac{t^2}{2n}$$

$$M_{\hat{S}_n}(t) \approx \left[1 + \frac{t^2}{2n} \right]^n$$

$$\approx \left[1 + \frac{t^2/2}{n} \right]^n \approx e^{t^2/2}$$

CALC : $\lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = e^x$

WE'VE SKETCHED:

$$\lim_{n \to \infty} M_{\hat{S}_n}(t) = e^{t^2/2}$$

$$\Rightarrow \quad \hat{S}_n \xrightarrow{d} N(0,1)$$

$\blacksquare$

PLAN FOR NEXT WEEK:

MONDAY — JUNETEENTH (NO CLASS)

TUESDAY — FINAL REVIEW (3H -CLASS)

WEDNESDAY — (OFFICE HOUR / CANCELLED / FUN STUFF)

THURSDAY — EXAM !!