MATH 201 (SUMMER 2023, SESH A2)

LECTURE 2 : 05/16/23

ANURAG SAHAY
OFF HRS: BY APPT (VIA ZOOM)

LECTURES :
9:00 AM – 11:15 AM (ET)
M, T, W, R

email: anuragsahay@rochester.edu

Zoom ID :
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. LECTURE 1 IS UPLOADED. (PANOPTO / SCHEDULE)
2. UPCOMING DEADLINES :
   - (a) WW 1 (WED, MAY 17th AT 11 PM ET)
   - (b) HW 1 (FRI, MAY 19th AT 11 PM ET)

     } OPEN
   - (c) WW 2 (FRI, MAY 19th AT 11 PM ET) } WILL OPEN TODAY
3. OFFICE HOURS THIS WEEK : WED & THURS, 4-5 PM ET

   (OR BY APPT.)
4. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!

RECALL

§ 1.1 SAMPLE SPACES & PROBABILITIES (CONTD.)

MATHEMATICAL MODEL FOR RANDOMNESS

↳ KOLMOGOROV'S AXIOMS

3 INGREDIENTS : $(\Omega, \mathcal{F}, P)$ PROBABILITY SPACE

Ω → SAMPLE SPACE

$\mathcal{F}$ → SET OF EVENTS

P → PROBABILITY (LIKELIHOOD)

$0 \leq P(A) \leq 1$

INTERLUDE : SETS & COUNTING

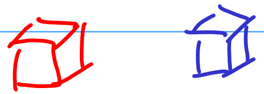
CARTESIAN PRODUCT

$(x_1, \dots, x_n)$ $\quad x_j \in A_j$

$$\underset{\substack{\uparrow\\ x_1}}{A_1} \times \underset{\substack{\uparrow\\ x_2}}{A_2} \times \cdots \times \underset{\substack{\uparrow\\ x_n}}{A_n} = \{ \underbrace{(x_1, \dots, x_n)} : x_j \in A_j \text{ FOR } j = 1, \dots, n \}$$

e.g. ROLLING TWO DICE (e.g. MONOPOLY)

$n = 2$, $A_1 = A_2 = \{1, 2, 3, 4, 5, 6\}$



(i, j)

$$\Omega = \{ (i,j) : i, j \in \{1, \dots, 6\} \}$$

$$= \{1, \dots, 6\} \times \{1, \dots, 6\} = A_1 \times A_2$$

Fact C.5. Let $A_1, A_2, \ldots, A_n$ be finite sets.

$$\#\left(A_1 \times A_2 \times \cdots \times A_n\right) = (\#A_1)\cdot(\#A_2)\cdots(\#A_n) = \prod_{i=1}^{n}(\#A_i).$$

$$\Omega = A_1 \times A_2 \qquad\qquad A_j = \{1, \ldots, 6\}$$

$$\#A_j = 6$$

$$6\left\{\underbrace{\begin{array}{cccc} (1,1) & (2,1) & \cdots & (6,1) \\ (1,2) & & & (6,2) \\ (1,3) & \vdots & \cdots & \vdots \\ (1,4) & & \ddots & \vdots \\ (1,5) & & & \vdots \\ (1,6) & \cdots & \cdots & (6,6) \end{array}}_{6}\right.$$

$$\#\Omega = 36 = \underbrace{6}_{\#A_1} \cdot \underbrace{6}_{\#A_2}.$$

Fact C.5. Let $A_1, A_2, \ldots, A_n$ be finite sets.

$$\#(A_1 \times A_2 \times \cdots \times A_n) = (\#A_1) \cdot (\#A_2) \cdots (\#A_n) = \prod_{i=1}^{n} (\#A_i).$$

PF: FOLLOWS FROM LAW OF MULT.

$$(\underbrace{x_1}_{\#A_1}, \underbrace{x_2}_{\#A_2}, \underbrace{x_3}_{\#A_3}, \ldots, \underbrace{x_n}_{\#A_n}) \qquad x_j \in A_j$$

$\#$ OF SUCH TUPLES $= (\#A_1) \cdot (\#A_2) \cdots \cdot (\#A_n)$

Example C.6. In a certain country license plates have three letters followed by three digits. How many different license plates can we construct if the country's alphabet contains 26 letters?

$\underline{L}\ \underline{L}\ \underline{L}\ \underline{D}\ \underline{D}\ \underline{D}$

L = SET OF LETTERS $\qquad \# L = 26$

D = SET OF DIGITS $\qquad \# D = 10$

$= \{0, 1, \ldots, 9\}$

$$\text{PLATES} = L \times L \times L \times D \times D \times D$$

$$\#\text{PLATES} = (\# L)\cdot(\# L)\cdot(\# L)\cdot(\# D)\cdot(\# D)\cdot(\# D)$$

$$= 26^3 \times 10^3$$

$$A_1 = A_2 = \cdots = A_n = A$$

$$A^n = \underbrace{A \times \cdots \times A}_{n}$$

e.g. $\Omega = \{1, \dots, 6\} \times \{1, \dots, 6\} = \{1, \dots, 6\}^2$

TWO FAIR DICE. $\Omega = \{1, \dots, 6\}^2$

$= \{ (i, j) : 1 \le i, j \le 6 \}$

$\# \Omega = 36$

$P(i,j) = 1/36$

e.g. $P\left(\begin{array}{l} \text{BLUE ROLL IS 2} \\ \text{\& RED ROLL IS 5} \end{array} \right) = \dfrac{1}{36}$

(R, B)

$P(D) = 5/36$

$D = \{ \text{SUM OF DICE-ROLL IS 8} \} = \left\{ \begin{array}{l} (2,6), (3,5) \\ (6,2), (5,3) \\ (4,4) \end{array} \right\}$

$P(D) = P\left\{ \begin{array}{l} (2,6), (3,5), (6,2), \\ (5,3), (4,4) \end{array} \right\} = P(2,6) + P(3,5) + P(6,2) + P(5,3) + P(4,4)$

$= \frac{1}{36} + \dots + \frac{1}{36} = 5/36$

EX.

$\{2, \ldots, 12\} \rightarrow$ ALL POSSIBLE SUMS.

$P(\text{ROLL} = 8) = \frac{1}{11}$ X

$P(\boxed{8}) > P(2)$

$\parallel$ $\qquad\qquad$ $\parallel$

$4+4$ $\qquad\qquad$ $1+1$

$\parallel$

$5+3$

$\parallel$

$2+6$

$\vdots$

Example 1.7. We flip a fair coin three times. Let us encode the outcomes of the flips as 0 for heads and 1 for tails. Then each outcome of the experiment is a sequence of length three where each entry is 0 or 1:

$$\Omega = \{(0,0,0), (0,0,1), (0,1,0), \ldots, (1,1,0), (1,1,1)\}. \tag{1.4}$$

$$= \{0,1\}^3 = \{0,1\} \times \{0,1\} \times \{0,1\}$$

(x_1, x_2, x_3)

$x_i \in \{0,1\}$

$$\# \Omega = [\# \{0,1\}]^3 = 2^3 = 8$$

$\omega \in \Omega$

$$P(\omega) = 1/8$$

$$P(\text{1st FLIP IS HEADS \& 3rd FLIP IS TAILS}) = P\{(0,0,1),(0,1,1)\}$$

$$= P(0,0,1) + P(0,1,1)$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

$(0, \underset{}{\overset{0/1}{\sim}}, 1)$

§ 1.2 RANDOM SAMPLING

$\Omega \rightarrow$ FINITE SAMPLE SPACE

"EQUALLY LIKELY OUTCOMES" $\Rightarrow$ FOR ANY $\omega \in \Omega$

$$\boxed{P(\omega) = \frac{1}{\#\Omega}}$$

$\hookrightarrow$ UNIFORM DIST.

e.g. FAIR DIE

FAIR COIN

SHUFFLED CARDS

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_n\} \qquad (\#\Omega = n)$$

$$P(\omega_1) = P(\omega_2) = \cdots = P(\omega_n)$$

$$\underline{\underline{1}} = P(\Omega) = P\{\omega_1, \dots, \omega_n\}$$

$$= P(\omega_1) + P(\omega_2) + \cdots + P(\omega_n)$$

$$= \underbrace{P(\omega_1) + \cdots + P(\omega_1)}_{n \text{ TIMES}}$$

$$= \underline{\underline{n P(\omega_1)}}$$

$$P(\omega_1) = \frac{1}{n} = \frac{1}{\#\Omega}$$

E.L.O.

$$A \subseteq \Omega \quad , \quad A = \{a_1, \ldots, a_n\}$$

$$P(A) = P\{a_1, \ldots, a_n\}$$

$$= P(a_1) + P(a_2) + \cdots + P(a_n)$$

$$P(\omega) = \frac{1}{\#\Omega}$$

$$P(A) = \frac{1}{\#\Omega} + \frac{1}{\#\Omega} + (n \text{ TIMES}) + \frac{1}{\#\Omega} = \frac{n}{\#\Omega} = \frac{\#A}{\#\Omega}$$

Fact 1.8. If the sample space Ω has finitely many elements and each outcome is equally likely then for any event $A \subset \Omega$ we have

$$P(A) = \frac{\#A}{\#\Omega}. \tag{1.5}$$

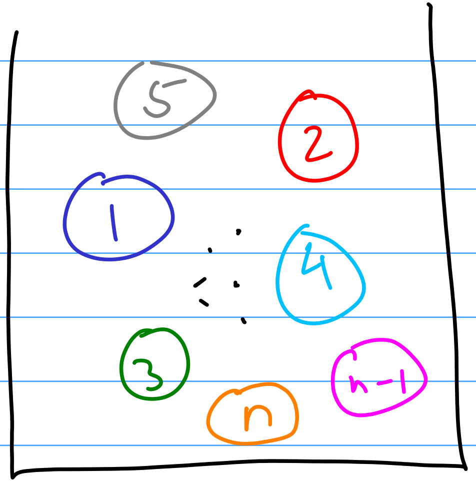
UNIFORM DISTRIBUTION

$\#A \rightarrow$ NUMBER OF FAVORABLE OUTCOMES

$\#\Omega \rightarrow$ TOTAL NUMBER OF OUTCOMES

N.B.: ALL OUTCOMES HAVE TO BE EQUALLY LIKELY.

SAMPLING
BALLS
FROM
AN
URN.



$$P(\text{PARTICULAR BALL}) = \frac{1}{\#\text{BALLS}}$$

$$S = \{1, 2, \ldots, n\}$$

I) SAMPLING WITH REPLACEMENT, ORDER MATTERS

OF BALLS

OF SAMPLES

$$S = \{1, 2, \ldots, \boxed{n}\}$$

$$\Omega = \underbrace{S \times \cdots \times S}_{k} = S^{\boxed{k}} = \{(s_1, \ldots, s_k) : s_j \in S\}$$

n n n

$$\# \Omega = \underbrace{n \times \cdots \times n}_{k} = n^k$$

$$\therefore P(\omega) = \frac{1}{\# \Omega} = \frac{1}{n^k}$$

$n = 5$
$k = 3$

Example 1.10. Suppose our urn contains 5 balls labeled $\{1, 2, 3, 4, 5\}$. Sample 3 balls with replacement and produce an ordered list of the numbers drawn. At each step we have the same 5 choices. The sample space is

$$\Omega = \{1, 2, 3, 4, 5\}^3 = \{(s_1, s_2, s_3) : \text{each } s_i \in \{1, 2, 3, 4, 5\}\}$$

and $\#\Omega = 5^3$. Since all outcomes are equally likely, we have for example

$$P\{\text{the sample is } (2,1,5)\} = P\{\text{the sample is } (2,2,3)\} = 5^{-3} = \frac{1}{125}. \qquad \blacktriangle$$

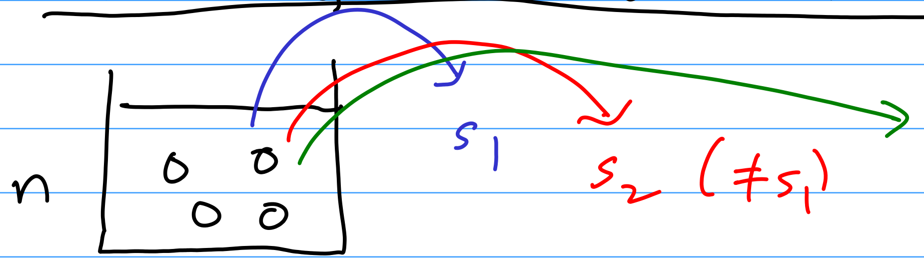
Q: WHAT PREVIOUSLY CONSIDERED SITUATIONS ARE SIMILAR TO THIS?

A1: ROLLING TWO DICE $\equiv$ SAMPLING TWO TIMES (W/ REPL, ORDER MATTERS) FROM 6 BALLS.

A2: FLIPPING 3 COINS
$\equiv$ $k = 3$, $n = 2$

n BALLS
k SAMPLES

SAMPLING WITHOUT REPLACEMENT, ORDER MATTERS



$(s_1, s_2, s_3) \neq (s_2, s_1, s_3)$

$S = \{1, \ldots, n\}$

$$\Omega = \{ (s_1, \ldots, s_k) : s_j \in S, \; s_i \neq s_j \text{ IF } i \neq j \}$$

$\# \Omega = ???$ [SEE NEXT FEW SLIDES]

COUNTING INTERLUDE: SAMPLING W/O REPETITION

Fact C.9. Consider all k-tuples $(a_1, \ldots, a_k)$ that can be constructed from a set A of size n $(n \geq k)$ without repetition. So each $a_i \in A$ and $a_i \neq a_j$ if $i \neq j$. The total number of these k-tuples is

$$\underbrace{(n)_k}_{\text{FALLING FACTORIAL}} = \underbrace{n \cdot (n-1) \cdots (n-k+1)}_{k \text{ TERMS IN PRODUCT}} = \frac{n!}{(n-k)!}.$$

$$l! = 1 \times 2 \times \cdots \times l$$

$$(n)_k = \frac{n(n-1)\cdots(n-k+1)(n-k)(n-k-1)\cdots(1)}{(n-k)(n-k-1)\cdots(1)}$$

$$= n! / (n-k)!$$

$\# S = n$ $\qquad s_i \neq s_j$

Pf

$$(\underbrace{s_1}_{n}, \underbrace{s_2}_{n-1}, \underbrace{s_3}_{n-2}, \dots, \underbrace{s_k}_{n-k+1})$$

AT $s_j \to n - j + 1$

$$\#\text{ OF SUCH TUPLES} = n \cdot (n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

Example C.10. I have a 5-day vacation from Monday to Friday in Santa Barbara. I have to specifically assign one day to exploring the town, one day to hiking the mountains, and one day for the beach. I can take up at most one activity per day. In how many ways can I schedule these activities?

$n = 5$

$$(\underbrace{D_1, D_2, D_3}_{k=3}) \qquad D_j \in \{ M, T, W, R, F \}$$

$T =$ TUES $\qquad R =$ THURS

$D_1 \rightarrow$ TOWN

$D_2 \rightarrow$ MOUNTAINS

$D_3 \rightarrow$ BEACH

$D_i \neq D_j$ IF $i \neq j$

$$\# \text{ OF WAYS} = (n)_k = (5)_3 = 5 \times 4 \times 3$$

$$= 60$$

In particular, with $k = n$, each n-tuple is an ordering or a $\boxed{permutation}$ of the set A. So the total number of orderings of a set of n elements is $n! = n \cdot (n-1) \cdots 2 \cdot 1$.

$$A = \{a_1, \dots, a_n\}$$

$(k = n)$ $\quad (s_1, \dots, s_n) \quad s_j \in A, \quad s_i \neq s_j$ IF $i \neq j$

$(a_1, \dots, a_n)$ $\quad$ # OF WAYS $= (n)_n = n(n-1) \cdots \cdot 1$
$= n!$

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

CONVENTION : $0! = 1$

OF WAYS TO REARRANGE AN EMPTY LIST.

$$(n)_k = \frac{n!}{(n-k)!}$$

$$(n)_n = n! / \boxed{0!} = 1$$

SKIP

SEE Book

Example C.11. Consider a round table with 8 seats.

(a) In how many ways can we seat 8 guests around the table?

(b) In how many ways can we do this if we do not differentiate between seating arrangements that are rotations of each other?

RETURN TO: SAMPLING WITHOUT REPLACEMENT, ORDER MATTERS

$$\Omega = \{ (s_1, \dots, s_k) : s_j \in S, \quad s_i \neq s_j \text{ IF } i \neq j \}$$

$$\# \Omega = (n)_k = n(n-1) \cdots (n-k+1) = \frac{n!}{(n-k)!}$$

$$P(\omega) = \frac{1}{\# \Omega} = \frac{(n-k)!}{n!} = \frac{1}{n(n-1) \cdots (n-k+1)}$$

$n = 5$ $k = 3$

Example 1.11. Consider again the urn with 5 balls labeled $\{1, 2, 3, 4, 5\}$ Sample $\boxed{3}$ balls without replacement and produce an ordered list of the numbers drawn. Now the sample space is

$$\Omega = \{(s_1, s_2, s_3) : \text{each } s_i \in \{1, 2, 3, 4, 5\} \text{ and } s_1, s_2, s_3 \text{ are all distinct}\}.$$

The first ball can be chosen in 5 ways, the second ball in 4 ways, and the third ball in 3 ways. So

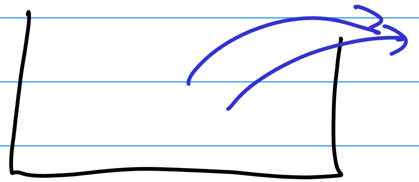
$$P\{\text{the sample is } \overset{\omega}{(2,1,5)}\} = \frac{1}{5 \cdot 4 \cdot 3} = \frac{1}{60}.$$

The outcome $\boxed{(2, 2, 3)}$ is not possible because repetition is not allowed. ▲

RESTART AT
10 : 21 AM

$S = \{1, \dots, n\} \supseteq \{s_1, \dots, s_k\}$

SAMPLING WITHOUT REPLACEMENT, ORDER IRRELEVANT



k BALLS OUT SIMULTANEOUSLY

N.B. ( ) vs { }

$(s_1, \dots, s_k) \neq (s_2, s_1, s_3, \dots, s_k)$

$\{s_1, \dots, s_k\} = \{s_2, s_3, s_1, \dots, s_k\}$

$\Omega = \{\omega \subseteq S : \#\omega = k\} \to$ SET OF ALL k-ELEMENT SUBSETS

$\#\Omega = ?$

$S \to \#S = n$

$\#\{A \subseteq S : \#A = k\}$

COUNTING INTERLUDE

Fact C.12. Let n and k be nonnegative integers with $0 \le k \le n$. The number of distinct subsets of size k that a set of size n has is given by the **binomial coefficient**

n CHOOSE k $\longleftarrow$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Pf. $f : A \xrightarrow[\ell\text{-TO-}1]{} B \qquad \#A = \ell \cdot \#B$

$A = \{(s_1, \dots, s_k) : s_i \neq s_j \text{ IF } i \neq j,\ s_j \in S\}$

$B = \{\{s_1, \dots, s_k\} : s_i \neq s_j \text{ IF } i \neq j,\ s_j \in S\}$

$$\# A = (n)_k = \frac{n!}{(n-k)!}$$

$$f : A \longrightarrow B$$

$$(s_1, \ldots, s_k) \longrightarrow \{s_1, \ldots s_k\}$$

EVERY REARRANGEMENT OF $\{s_1, \ldots, s_k\}$ AS A TUPLE

$= k!$

$f \to k!$ - TO - 1 [i.e. $l = k!$]

$$\# A = l \cdot \# B$$

$$\Rightarrow \frac{n!}{(n-k)!} = k! \cdot \# B$$

$$\Rightarrow \# B = \frac{n!}{(n-k)!\, k!} = \binom{n}{k}$$

Example C.14. In a class there are 12 boys and 14 girls. How many different teams of 7 pupils with 3 boys and 4 girls can be created?

\# BOYS

① HOW MANY WAYS TO CHOOSE 3 BOYS $= \binom{12}{3}$

\# OF BOYS TO BE CHOSEN

\# GIRLS

② HOW MANY WAYS TO CHOOSE 4 GIRLS $= \binom{14}{4}$

\# TO BE CHOSEN

$T_1 \to$ BOYS

$T_2 \to$ GIRLS

$$(\underbrace{T_1}_{\binom{12}{3}}, \underbrace{T_2}_{\binom{14}{4}})$$

TOTAL # OF CHOICES $= \binom{12}{3}\binom{14}{4}$

RETURN TO: SAMPLING WITHOUT REPLACEMENT, ORDER IRRELEVANT

$$\Omega = \{ \omega \subseteq S : \# \omega = k \}$$ → SET OF ALL k-ELEMENT SUBSETS

$$\# \Omega = \binom{n}{k}$$

$$P(\omega) = \frac{1}{\# \Omega} = \frac{1}{\binom{n}{k}} = \frac{(n-k)! \cdot k!}{n!}$$

ANOTHER WAY: INTRODUCE ORDER

$$\Omega_1 = \{ (s_1, \dots, s_k) : s_i \neq s_j, s_j \in S \}$$

$\# \Omega_1 = (n)_k$

$$P(\{s_1, \dots, s_k\}) = \frac{\# \text{ OF FAVORABLE OUTCOMES}}{\text{TOTAL } \# \text{ OF OUTCOMES}}$$

FAVORABLE OUTCOME $\longrightarrow$ PERMUTATION OF $(s_1, \dots, s_k)$

$k!$

$$P(\{s_1, \dots, s_k\}) = \frac{k!}{(n)_k} = \frac{(n-k)! \cdot k!}{n!}$$

$\boxed{\text{SAME ANSWER}}$

SAME ANSWER !

→ MANY COUNTING MODELS CAN APPLY TO THE SAME PROBALISTIC PROBLEM

WARNING : DON'T MIX MODELS
(JUST LIKE ALCOHOL)

→ SOMETIMES ADDING STRUCTURE SIMPLIFIES THE PROBLEM.

$n=5$ $k=3$

Example 1.12. Suppose our urn contains 5 balls labeled $\underbrace{1, 2, 3, 4, 5}$. Sample $\underbrace{3}$ balls without replacement and produce an unordered set of 3 numbers as the outcome. The sample space is

$$\Omega = \{\omega : \omega \text{ is a 3-element subset of } \{1,2,3,4,5\}\}.$$

For example

$$\frac{2 \times 6}{120}$$

$$P(\text{the sample is } \{1,2,5\}) = \frac{1}{\binom{5}{3}} = \frac{2!3!}{5!} = \frac{1}{10}.$$

The outcome $\{2, 2, 3\}$ does not make sense as a set of three numbers because of the repetition. ▲

SKIP (FOR NOW)

Example 1.13. Suppose we have a class of 24 children. We consider three different scenarios that each involve choosing three children.

(a) Every day a random student is chosen to lead the class to lunch, without regard to previous choices. What is the probability that Cassidy was chosen on Monday and Wednesday, and Aaron on Tuesday?

(b) Three students are chosen randomly to be class president, vice president, and treasurer. No student can hold more than one office. What is the probability that Mary is president, Cory is vice president, and Matt treasurer?

[ALSO: P(BEN IS PRESIDENT)?]

(c) A team of three children is chosen at random. What is the probability that the team consists of Shane, Heather and Laura?

[ALSO: P(MARY IS ON THE TEAM)?]

Example 1.14. Our urn contains 10 marbles numbered 1 to 10. We sample 2 marbles without replacement. What is the probability that our sample contains the marble labeled 1? Let A be the event that this happens. However we choose to count, the final answer $P(A)$ will come from formula (1.5).

ORDER DOESN'T MATTER

$(10)_2 = 90$

ORDER MATTERS

$$\Omega = \{ (m_1, m_2) : m_j \in \{1, \dots, 10\}, m_1 \neq m_2 \}$$

$$A = \{ (m_1, m_2) : \text{EITHER } m_1 = 1, m_2 \neq 1 \text{ OR } m_1 \neq 1, m_2 = 1 \}$$

$$P(A) = P(\{(1, m_2) : m_2 \neq 1\} \cup \{(m_1, 1) : m_1 \neq 1\}) = P\{(1, m_2) : m_2 \neq 1\} + P\{(m_1, 1) : m_1 \neq 1\}$$

$$= \frac{\#\text{ FAVORABLE OUTCOMES}}{\#\Omega}$$

$$\frac{9}{90} + \frac{9}{90} = \frac{1}{5}$$

ORDER DOESN'T MATTER

$$\Omega = \{ \{m_1, m_2\} \subseteq \{1, \dots, 10\} : m_1 \neq m_2 \}$$

$$\# \Omega = \binom{10}{2} = \frac{10 \cdot 9}{2} = 45$$

$$A = \{ \{1, m\} : m \neq 1 \}$$

$$\# A = 9$$

$$P(A) = \frac{\# A}{\# \Omega} = \frac{9}{45} = \frac{1}{5}$$

LESSON: IF ALL OUTCOMES ARE EQUALLY LIKELY, AND E IS AN EVENT IN WHICH ORDER DOESN'T MATTER, THEN P(E) CAN BE COMPUTED BY EITHER MODEL. (w/o REPLACEMENT)

§ 1.3 INFINITELY MANY OUTCOMES

WE NOW CONSIDER THE CASE WHERE

$$\# \Omega = \infty$$

e.g. KEEP FLIPPING A COIN UNTIL IT TURNS UP HEADS

(TTTT...)

$$\underbrace{H}_{1},\ \underbrace{TH}_{2},\ \underbrace{TTH}_{3},\ TTTH,\ TTTTH,\ \ldots$$

$$\Omega = \{\underbrace{\infty}_{\text{no head occurs.}}, 1, 2, 3, \ldots, \underbrace{k}_{\text{where H occurs}}, \ldots\}$$

TEST YOUR INTUITION

IF THE COIN IS FAIR,

$$P(\infty) = 0$$

$$P(k) = P(\underbrace{TT\cdots T}_{k-1}\underbrace{H}_{k\text{th PLACE}}) = \frac{\text{\# OF FAVORABLE OUTCOMES}}{\text{\# OF TOTAL OUTCOMES}}$$

$$= \frac{1}{2^k}$$

ALSO $\Omega = \{\infty, 1, 2, \ldots\}$

RECALL:

$$P\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j=1}^{\infty} P(A_j)$$

DISJOINT

$$\Omega = \{\infty\} \cup \{1\} \cup \{2\} \cup \cdots$$

$$= \bigcup_{j=0}^{\infty} A_j$$ DISJOINT

$$A_0 = \{\infty\}, \qquad A_j = \{j\} \quad (\text{IF } j \geq 1)$$

$$\underbrace{P(\Omega)}_{1} = P\left(\bigcup_j A_j\right) = \sum_{j=0}^{\infty} P(A_j)$$

$$= P(\infty) + \sum_{j=1}^{\infty} P(j)$$

$$= P(\infty) + \boxed{\sum_{j=1}^{\infty} 2^{-j}} \to 1$$

$$\sum_{j=1}^{\infty} 2^{-j} = \frac{1/2}{1-1/2} = 1$$

$$1 = P(\infty) + 1 \Rightarrow P(\infty) = 0.$$

<u>CHECK</u>: $H \to p \qquad (p \in [0,1])$

$T \to 1-p$

1. $P(\infty) = 0$ IF $p > 0$.
2. $P(\infty) = 1$ IF $p = 0$.