MATH 201 (SUMMER 2023, SESH A2)

LECTURE 3 : 05/17/23

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LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. LECTURE 2 (INCLUDING SOME EXTRA) IS UPLOADED. (PANOPTO / SCHEDULE)
2. UPCOMING DEADLINES:
   a. WW 1 (WED, MAY 17th AT 11 PM ET) — TODAY
   b. HW 1 (FRI, MAY 19th AT 11 PM ET) } OPEN
   c. WW 2 (~~FRI, MAY 19th~~ SAT, MAY 20th AT 11 PM ET) } WILL OPEN TODAY
3. OFFICE HOURS THIS WEEK: TODAY & THURS, 4-5 PM ET. (OR BY APPT.)
4. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!

§ 1.3 INFINITELY MANY OUTCOMES (CONT'D.)

$\# \Omega = \infty$

$\boxed{P(A) = \frac{\# A}{\# \Omega}}$

UNIFORM PROBABILITY IN $\mathbb{R}$

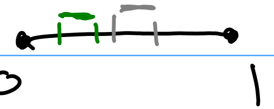
Example 1.17. We pick a real number uniformly at random from the closed unit interval $[0, 1]$. Let X denote the number chosen. “Uniformly at random” means that X is equally likely to lie anywhere in $[0, 1]$. Obviously $\Omega = [0, 1]$. What is the probability that X lies in a smaller interval $[a, b] \subseteq [0, 1]$? Since all locations for X are equally likely, it appears reasonable to stipulate that the probability that X is in $[a, b]$ should equal the proportion of $[0, 1]$ covered by $[a, b]$:

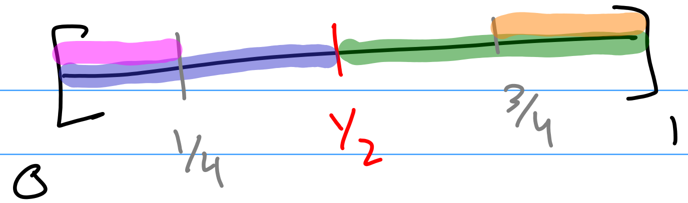
UNIFORM PROBABILITY.

$$\boxed{P\{X \text{ lies in the interval } [a, b]\} = b - a \quad \text{for } 0 \leq a \leq b \leq 1.} \tag{1.11}$$

$\Omega = [0,1]$

$P(\in [a,b]) = b - a.$





$P \propto \text{LENGTH}$ $\qquad \Omega = [0,1]$

$P(\Omega) = 1$

$$\frac{P(I)}{\underset{\overset{\shortparallel}{1}}{P(\Omega)}} = \frac{\text{LENGTH}(I)}{1}$$

IN GENERAL, IF THE SAMPLE SPACE Ω IS ENDOWED WITH A NATURAL MEASURE OF SIZE

(e.g. AREA, CARDINALITY, VOLUME ...)

THEN A UNIFORM SAMPLE SHOULD SATISFY

$P(A) \propto \text{SIZE}(A)$ FOR EVENTS $A \subseteq \Omega$

$$\therefore \frac{P(A)}{\boxed{P(\Omega)}} = \frac{\text{SIZE}(A)}{\text{SIZE}(\Omega)} \Rightarrow P(A) = \frac{\text{SIZE}(A)}{\boxed{\text{SIZE}(\Omega)}}$$

1

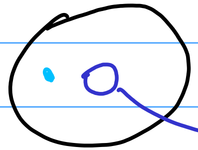
$\in (0, \infty)$

UNIFORM

$$P(A) = \frac{SIZE(A)}{SIZE(\Omega)}$$

Example 1.18. Consider a dartboard in the shape of a disk with a radius of 9 inches. The bullseye is a disk of diameter $\frac{1}{2}$ inch in the middle of the board. What is the

$A \subseteq$



$R = 9$

$r = 1/4$

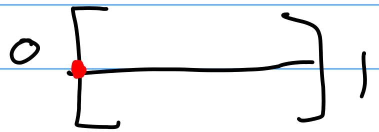
probability that a dart randomly thrown on the board hits the bullseye?

$$AREA(\Omega) = \pi R^2 = \pi \cdot (9^2) = 81\pi$$

$$P(\text{HITTING } \underbrace{\text{BULLSEYE}}_{A}) = P(A) = \frac{AREA(A)}{AREA(\Omega)} = \frac{\pi (1/4)^2}{81\pi} = \frac{1}{16} \cdot \frac{1}{81}$$

Q. WHAT IS $P(0)$ WHEN SAMPLING FROM $[0,1]$ UNIFORMLY?

$P([a,b]) = b - a$



$\{0\} = [0,0]$

$P(\{0\}) = P([0,0])$

$= 0 - 0 = 0$

$[a,b] = \{a \le x \le b\}$

$x \le 0$ & $x \geq 0$

$\Rightarrow P(x) = 0 \quad \forall x$

$P(\{x\}) = P([x,x]) = x - x = 0$

① $P(\overbrace{[0,1]}^{\Omega}) = 1$

② $P(x) = 0$ FOR EVERY $x \in [0,1]$

③ $[0,1] = \bigcup_{x \in [0,1]} \{x\}$

$\boxed{④}$ BY ADDITIVITY $P(\underbrace{\cup A_j}_{\text{DISJOINT.}}) = \sum P(A_j)$

$1 \overset{①}{=} P([0,1]) \overset{③}{=} P\left(\bigcup_{x \in [0,1]} \{x\}\right) \boxed{\overset{④}{=}} \sum_{x \in [0,1]} P(x) \overset{②}{=} \sum_{x \in [0,1]} 0 = 0$

CONTRADICTION ?

COUNTABLE

$A_1, A_2, A_3, \ldots$ (DISJOINT)

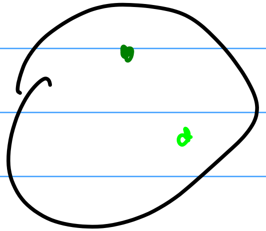
$$P\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} P(A_j)$$

$$\boxed{\{x\} \longrightarrow x \in [0,1]}$$

$\to$ UNCOUNTABLE.

$[0,1]$ IS UNCOUNTABLE

$(\mathbb{R})$



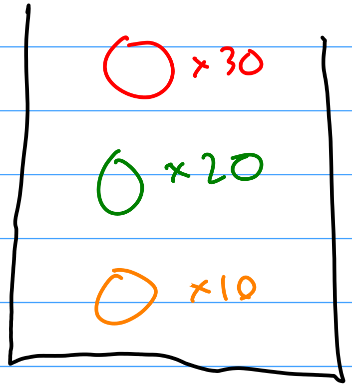
§ 1.4 CONSEQUENCES OF THE RULES OF PROBABILITY

IDEA 1: TO UNDERSTAND A COMPLICATED EVENT, DECOMPOSE IT IN TERMS OF SIMPLER AND MUTUALLY EXCLUSIVE EVENTS

$$P(A) = P\left(\bigcup_{j=1}^{\infty} A_j\right) = P(A_1) + P(A_2) + \cdots$$

$$A = \bigcup_{j=1}^{\infty} A_j$$

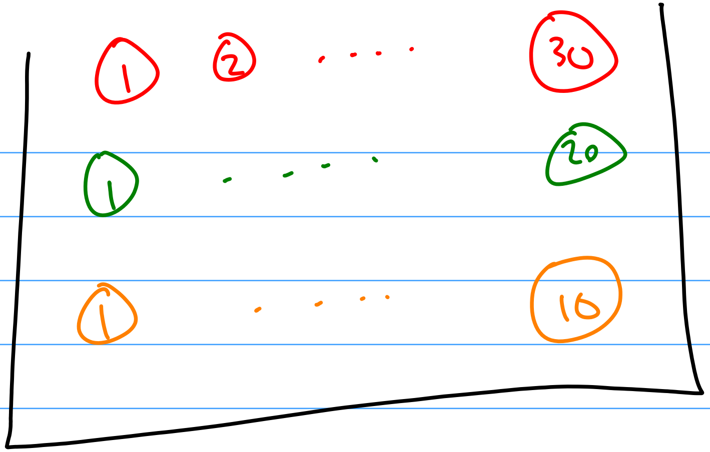
Example 1.19. An urn contains 30 red, 20 green and 10 yellow balls. Draw two without replacement. What is the probability that the sample contains exactly one red or exactly one yellow? To clarify the question, it means the probability that the sample contains exactly one red, or exactly one yellow, *or both* (inclusive or). This interpretation of *or* is consistent with unions of events.



P(CONTAINS EXACTLY ONE RED BALL OR EXACTLY ONE YELLOW BALL)

$$A = (\underbrace{RY}_{A_1}) \cup (\underbrace{RG}_{A_2}) \cup (\underbrace{YG}_{A_3})$$

$$P(A) = P(A_1) + P(A_2) + P(A_3)$$



ORDER DOESN'T MATTER.

$$\#\Omega = \binom{60}{2} = \frac{60\cdot 59}{2} = 30\times 59$$

$$P(A_1) = P(RY) = \boxed{\frac{\#\text{ OF FAVOR. OUTCOMES}}{\#\text{ OF TOTAL OUTCOMES}}}$$

$$= \frac{30\cdot 10}{30\times 59} = \frac{10}{59}$$

$$P(A_2) = P(RG) = \frac{30\cdot 20}{30\times 59} = \frac{20}{59}$$

$$P(A_3) = P(YG) = \frac{10\cdot 20}{30\cdot 59} = \frac{20}{177}$$

$$P(A) = P(A_1) + P(A_2) + P(A_3)$$

$$= \frac{10}{59} + \frac{20}{59} + \frac{20}{177}$$

$$\boxed{P(A) = \frac{110}{177}}$$

Example 1.20. Peter and Mary take turns rolling a fair die. If Peter rolls 1 or 2 he wins and the game stops. If Mary rolls 3, 4, 5, or 6, she wins and the game stops. They keep rolling in turn until one of them wins. Suppose Peter rolls first.

(a) What is the probability that Peter wins and rolls at most 4 times?

TRY THIS

$$\boxed{A_{\leq 4}} = A_1 \cup A_2 \cup A_3 \cup A_4$$

A_j = PETER WINS ON HIS jth ROLL

FAVORABLE

$$P(A_1) = P(\text{PETER ROLLED 1/2 IN THE FIRST ROLL}) = \frac{2}{6} = \frac{1}{3}$$

TOTAL

$$P(A_2) = P\left[\left(\overset{P}{\underset{}{\frac{3,4,5,6}{}}},\ \underset{M}{\frac{1,2}{}},\ \overset{P}{\frac{1,2}{}}\right)\right] = \frac{4 \cdot 2 \cdot 2}{6^3} = \frac{2}{27}$$

$$P(A_j) = P\Big(\underset{\mathrm{P}}{\underline{\overset{1}{R}}} \ \underset{\mathrm{M}}{\underline{\overset{1}{S}}} \ \underline{\overset{2}{R}} \ \underline{\overset{2}{S}} \ \ldots \ \underline{\overset{j-1}{R}} \ \underline{\overset{j-1}{S}} \ \underline{\overset{j}{S}} \Big)$$

$R = \{3, 4, 5, 6\}$

$S = \{1, 2\}$

$$= \frac{4^{j-1}\, 2^{j}}{6^{j} \cdot 6^{j-1}}$$

FAVORABLE OUTCOMES

TOTAL

$$= \frac{2^{j-1}}{3^{2j-1}}$$

$$P(A_1) = \frac{1}{3} = \frac{2^{1-1}}{3^{2\cdot 1-1}}$$

$$P(A_2) = \frac{2}{27} = \frac{2^{2-1}}{3^{2\cdot 2-1}}$$

$$P(A) = P(A_1) + P(A_2) + P(A_3) + P(A_4)$$

$$= \sum_{j=1}^{4} \frac{2^{j-1}}{3^{2j-1}} \longrightarrow \text{GEOMETRIC SERIES}$$

$$\left(a = \frac{1}{3},\ r = \frac{2}{9} \right)$$

(b) What is the probability that Mary wins?

$R = \{3, 4, 5, 6\}$

$S = \{1, 2\}$

$B \to$ MARY WINS.

$$B = \bigcup_{j=1}^{\infty} B_j$$

$B_j \to$ MARY WINS ON ROLL j

$$P(B) = \sum_{j=1}^{\infty} P(B_j)$$

$$P(B_j) = P\left(\underline{\overset{1}{R}}\ \underline{\overset{1}{S}}\ \underline{\overset{2}{R}}\ \underline{\overset{2}{S}} \ \cdots\ \underline{\overset{j-1}{S}}\ \underline{\overset{j}{R}}\ \underline{\overset{j}{R}} \right) = \frac{4^{j+1}\, 2^{j-1}}{\underbrace{6^j\, 6^j}_{\text{TOTAL}}}$$

P M

$$P(B_j) = \frac{2^{j+1}}{3^{2j}}$$

$$P(B) = \sum_{j=1}^{\infty} \frac{2^{j+1}}{3^{2j}}$$ GEOMETRIC

$a = 4/9$

$r = 2/9$

$$= \frac{4}{9} + \frac{4}{9}\left(\frac{2}{9}\right) + \frac{4}{9}\left(\frac{2}{9}\right)^2 + \frac{4}{9}\left(\frac{2}{9}\right)^3 + \cdots$$

$$= \frac{a}{1-r} = \frac{4/9}{1-2/9} = \frac{4/9}{7/9} = \boxed{\frac{4}{7}}$$

PROB. THAT MARY WINS.

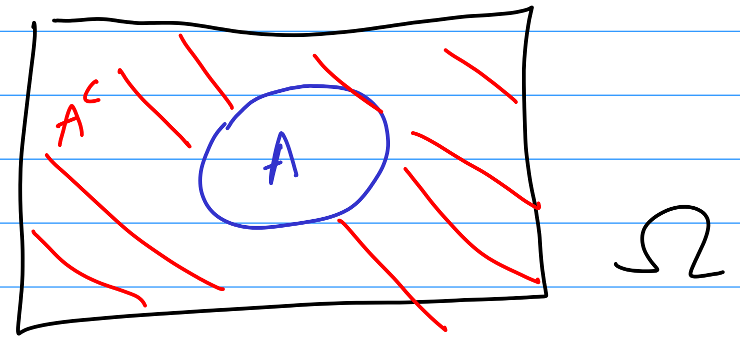
BACK AT

10 : 13

IDEA 2: IF A IS MORE COMPLEX THAN A^c, THEN STUDY A^c INSTEAD

$$\Omega = A \cup A^c$$

$$1 = P(\Omega) = P(A) + P(A^c)$$



$$\Rightarrow \quad P(A) = 1 - P(A^c)$$

Example 1.21. Roll a fair die 4 times. What is the probability that some number appears more than once? ~~A moment's thought reveals that this event contains~~

$$P\Big(\underbrace{\text{SOME NUMBER APPEARS MORE THAN ONCE}}_{E}\Big) = P(A_1) + \cdots + P(A_6)$$

REQUIRES DISJOINTNESS

A_j = j APPEARS MORE THAN ONCE

$$E = \bigcup_{j=1}^{6} A_j$$

$\boxed{2\ 2\ 4\ 4} \in E$

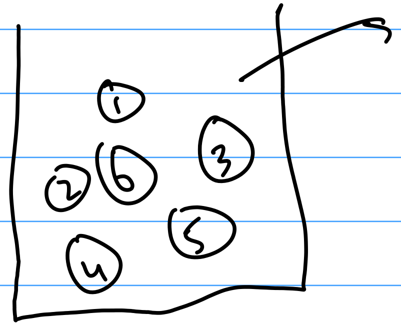
ROLL

A_2, A_4

$A_2 \cap A_4 \neq \emptyset$

E^c = NO NUMBER APPEARS MORE THAN ONCE.

= ALL FOUR ROLLS ARE DISTINCT.



SAMPLING W/O REPLACEMENT (ORDER MATTERS)

$n = 6$

$k = 4$

$$(6)_4 = 6 \times 5 \times 4 \times 3$$

$$P(E^c) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{6 \times 5 \times 4 \times 3}{6^4} = 5/18$$

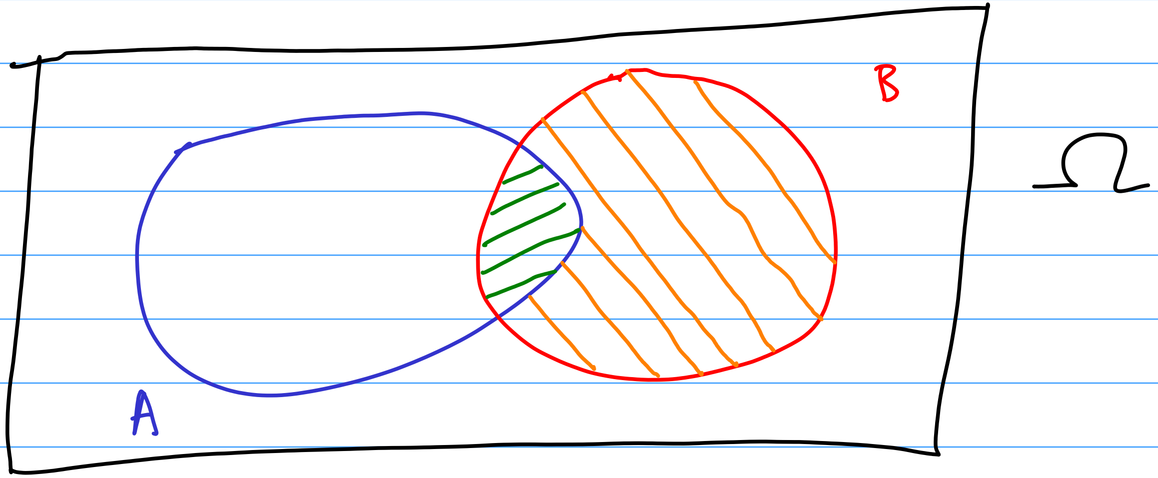
ANS.

$$P(E) = 1 - P(E^c) = 1 - 5/18 = \boxed{13/18}$$

USEFUL FACT:

$$P(B) = P(A \cap B) + P(A^c \cap B) \quad (= P(B \setminus A))$$

($B = \Omega$ gives pvs. fact)



IDEA 3 : PROBABILITY IS MONOTONIC.

i.e. IF $A \subseteq B$, THEN $P(A) \leq P(B)$ } → INTUITION

e.g. $\Omega \rightarrow$ DIE-ROLL $= \{1, 2, \ldots, 6\}$

B = OUTCOME IS EVEN

A = OUTCOME IS 2

$A \subseteq B$

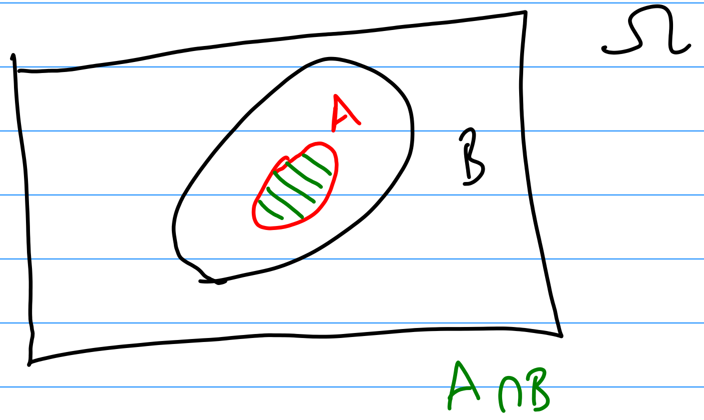
Pf : $P(B) = P(B \cap A) + \underbrace{P(B \setminus A)}_{\geq 0}$, $\quad B \setminus A = B \cap A^c$

$\Rightarrow \quad P(B) \geq P(B \cap A) = P(A)$

$A \subseteq B, \quad B \cap A = A$

$\Downarrow$

$P(A) \leq P(B)$



RECALL, FLIPPING A (FAIR) COIN TILL A HEAD SHOWS UP

$\Omega = \{ \infty, 1, \underset{\substack{\uparrow \\ TH}}{2}, \dots \}$ $\qquad \underbrace{T \cdots T}_{k-1} H \equiv k$

WE SHOWED

$P(\infty) = 0$ i.e. THE PROB. OF NEVER SEEING HEADS IS 0.

ALT. PROOF: SET A_k = FIRST k FLIPS ARE TAILS

$$A_k = \{k+1, k+2, \dots, \infty\}$$

$\{\infty\} \subseteq A_k$ FOR EVERY k.

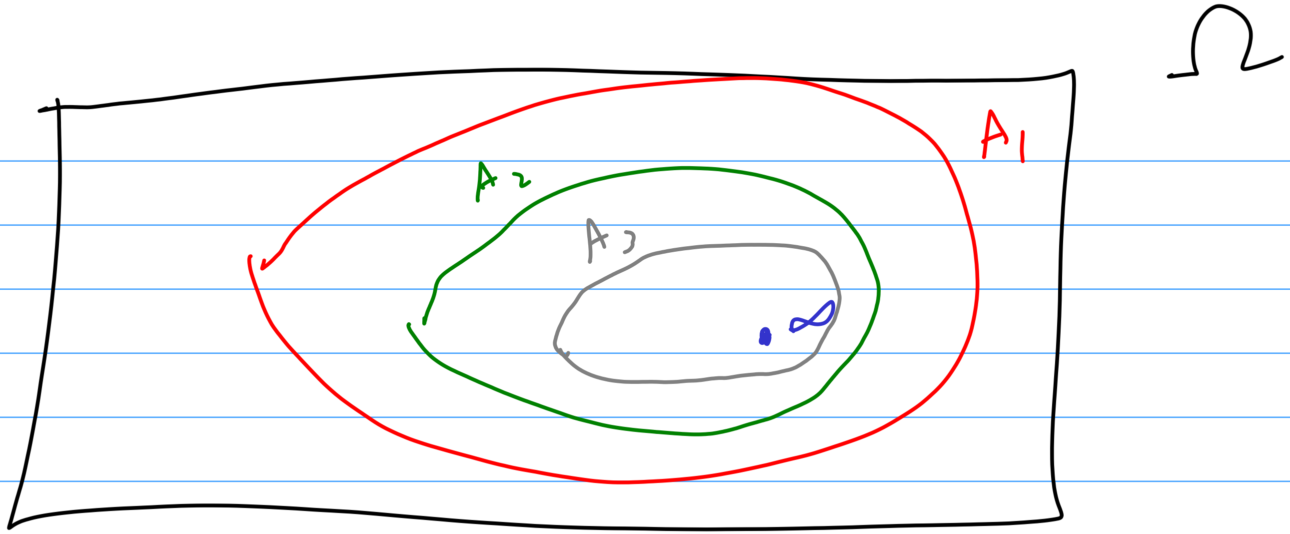
$$\underbrace{T \; T \; T \; T \; T}_{k} \; T$$

$\Rightarrow$ BY MONO.

$$P(\infty) \le P(A_k) = \frac{\#\text{FAVOR}}{\#\text{TOTAL}} = \frac{1}{2^k}$$

$0 \le P(\infty) \le 2^{-k}$ FOR EVERY k.

$\Rightarrow P(\infty) = 0$ (e.g. BY LIM $k \to \infty$)



SCHEMATIC FOR

PRVS ARGUMENT

IDEA 4: INCLUSION - EXCLUSISION

A, B, C NOT NEC. DISJOINT

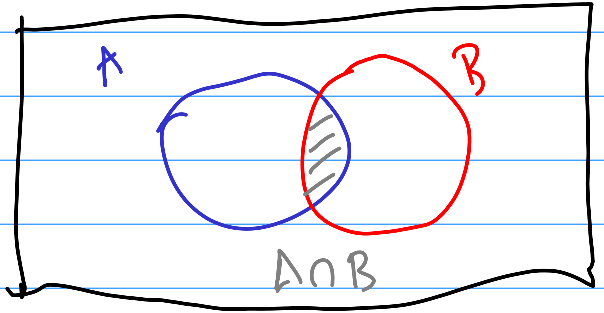
Fact 1.23. (Inclusion-exclusion formulas for two and three events)

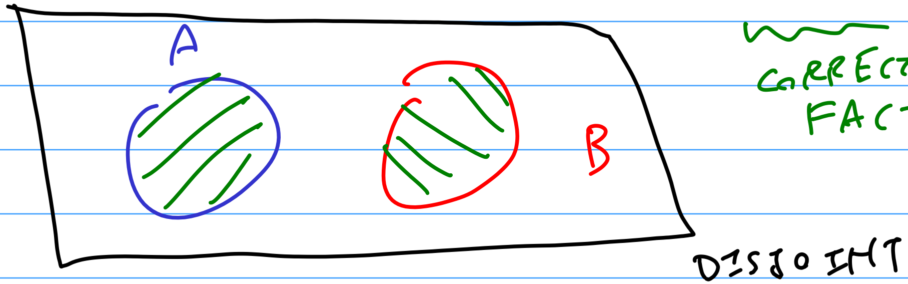
$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B).} \tag{1.16}$$

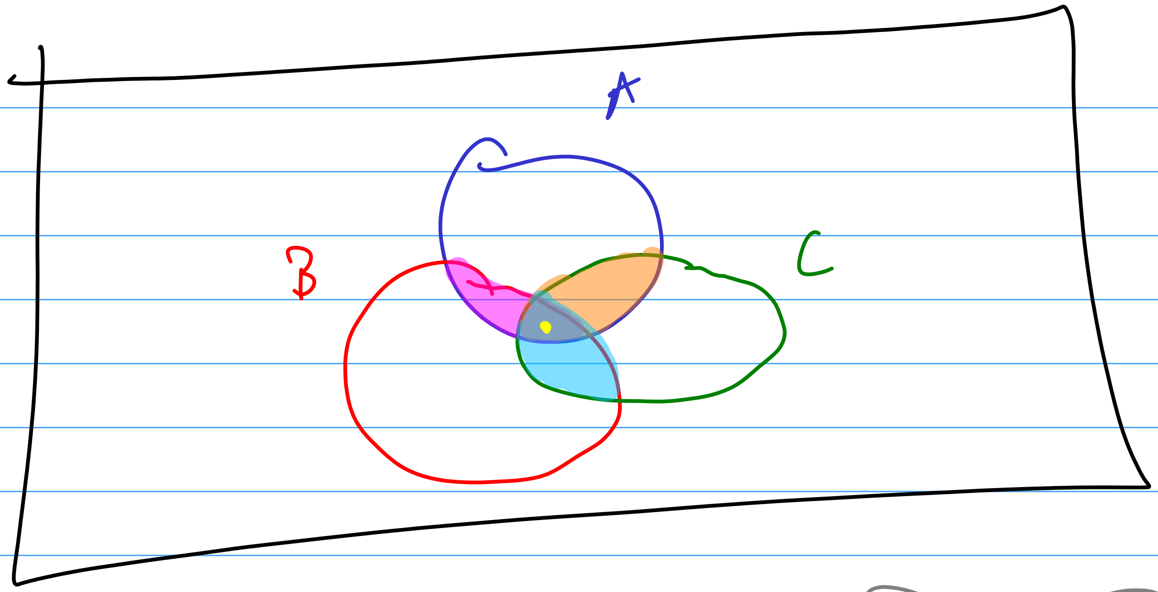
$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C). \tag{1.17}$$

Pf: VENN DIAGRAM

$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\text{CORRECTION FACTOR}}$



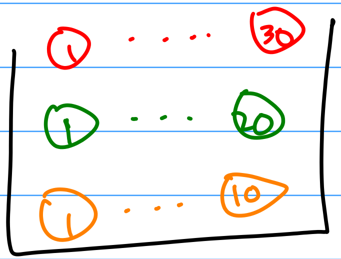




1st CORRECTION

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + \underbrace{P(A \cap B \cap C)}_{\text{2nd CORRECTION}}$$

Example 1.24. (Example 1.19 revisited) An urn contains 30 red, 20 green and 10 yellow balls. Draw two without replacement. What is the probability that the sample contains exactly one red or exactly one yellow?



$$A = A_1 \cup A_2 \cup A_3$$

RG YG RY

$B_1 \cap B_2$

$$A = B_1 \cup B_2$$

$B_1 = \{$EXACTLY ONE RED$\}$

$B_2 = \{$EXACTLY ONE YELLOW$\}$

$$\#\Omega = \binom{60}{2} = 30\cdot 59$$

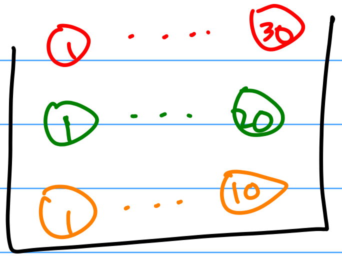
$$P(A) = P(B_1) + P(B_2) - \underbrace{P(B_1 \cap B_2)}_{\text{CORRECTION}}$$

\# FAVOR

$$P(B_1) = P(\text{EXACTLY ONE R}) = \frac{30\cdot \boxed{30}}{30\cdot 59}$$

$30 \rightarrow 20 + 10$

\# TOTAL



$$= \frac{30}{59}$$

$$P(B_2) = P(\text{EXACTLY ONE Y}) = \frac{10\cdot 50}{30\cdot 59} = \frac{50}{177}$$

$$P(B_1 \cap B_2) = P(\text{EXACTLY RY})$$

$$= \frac{30 \cdot 10}{30 \cdot 59} = \frac{10}{59}$$

$$P(A) = P(B_1) + P(B_2) - P(B_1 \cap B_2)$$

$$= \frac{30}{59} + \frac{50}{177} - \frac{10}{59} = \frac{110}{177} \quad //$$

(USES IDEA 2 & 4)

Example 1.25. In a town 15% of the population is blond, 25% of the population has blue eyes and 2% of the population is blond with blue eyes. What is the probability that a randomly chosen individual from the town is not blond and does not have blue eyes? (We assume that each individual has the same probability to be chosen.)

$P(\text{BLOND}) = \frac{15}{100}$ //

$P(\text{BLUE}) = \frac{25}{100}$ //

$P(\text{BLOND \& BLUE}) = \frac{2}{100}$

$A = \text{BLOND}$

$B = \text{BLUE}$

$P(\text{NOT BLOND \& NOT BLUE})$

$P(E) + P(E^c) = 1$

$= P(A^c \cap B^c) \underset{\text{de M}}{=} P((A \cup B)^c) \overset{?}{=} 1 - P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{15}{100} + \frac{25}{100} - \frac{2}{100} = \frac{38}{100}$$

$$P\left(\begin{array}{c}\text{NOT BLOND \&} \\ \text{NOT BLUE}\end{array}\right) = 1 - \frac{38}{100} = \frac{62}{100} = \frac{31}{50}$$

$$= 62\%$$

APPROXIMATION

Fact 1.26. (General inclusion-exclusion formula)

$$
\begin{aligned}
P(A_1 \cup \cdots \cup A_n) &= \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2}) \\
&\quad + \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) \\
&\quad - \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4}) \\
&\quad + \cdots + (-1)^{n+1} P(A_1 \cap \cdots \cap A_n) \\
&= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} P(A_{i_1} \cap \cdots \cap A_{i_k}). \qquad (1.20)
\end{aligned}
$$

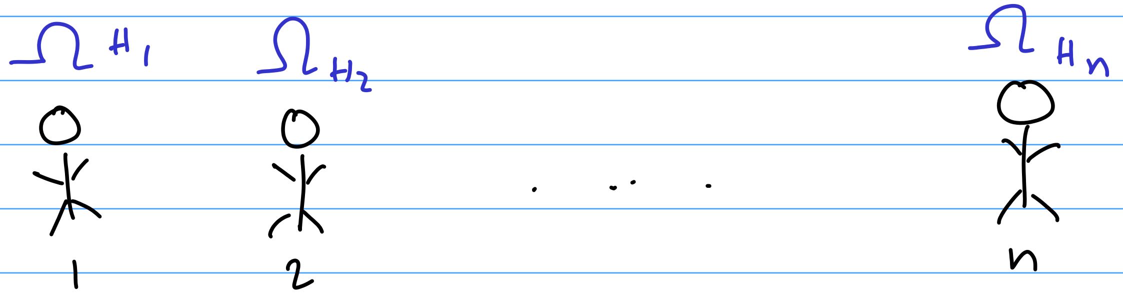
1st CORRECTION FACTOR

2nd

LAST FACTOR

INTERPRETATION?

Example 1.27. Suppose n people arrive for a show and leave their hats in the cloakroom. Unfortunately, the cloakroom attendant mixes up the hats completely so that each person leaves with a random hat. Let us assume that all $n!$ assignments of hats are equally likely. What is the probability that no one gets ~~his/her~~ THEIR own hat? How does this probability behave as $n \to \infty$?



A_j = PERSON j GET THEIR OWN HAT

A_j^c = PERSON j DOESN'T GET THEIR OWN HAT

$$E = \bigcap_{j=1}^{n} A_j^c \underset{\text{de M's LAW}}{=} \left(\bigcup_{j=1}^{n} A_j \right)^c$$

$$P(E) = P\left[\left(\bigcup_{j=1}^{n} A_j \right)^c \right] = 1 - \underbrace{P\left(\bigcup_{j=1}^{n} A_j \right)}$$

APPLY GENERAL FORM OF INCL./EXCL.

Fact 1.26. (General inclusion-exclusion formula)

$$\frac{(n-1)!}{n!} \qquad \frac{(n-2)!}{n!}$$

$$P(A_1 \cup \cdots \cup A_n) = \sum_{i=1}^{n} \boxed{P(A_i)} - \sum_{1 \le i_1 < i_2 \le n} \boxed{P(A_{i_1} \cap A_{i_2})}$$

$$+ \sum_{1 \le i_1 < i_2 < i_3 \le n} \boxed{P(A_{i_1} \cap A_{i_2} \cap A_{i_3})} \rightarrow \frac{(n-3)!}{n!}$$

$$- \sum_{1 \le i_1 < i_2 < i_3 < i_4 \le n} \boxed{P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4})}$$

$$+ \cdots + (-1)^{n+1} \boxed{P(A_1 \cap \cdots \cap A_n)}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \cdots < i_k \le n} \boxed{P(A_{i_1} \cap \cdots \cap A_{i_k}).} \qquad (1.20)$$

$$\frac{(n-k)!}{k!}$$

$$1 \leq i_1 < i_2 < i_3 \cdots < i_k \leq n$$

$$P\left(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cdots \cap A_{i_k}\right) = \frac{\#\ \text{FAVORABLE}}{\#\ \text{TOTAL}}$$

$$= \frac{(n-k)!}{n!}$$

$$\underset{1}{\_}\ \ \_\ \ \boxed{\underset{i_1}{\_}}\ \ \_\ \ \boxed{\underset{i_2}{\_}}\ \cdots\ \boxed{\underset{i_k}{\_}}\ \ \_\ \ \underset{n}{\_} \longrightarrow (n-k)!$$

k HATS ARE FIXED

$$\boxed{\sum_{1 \le i_1 < \cdots < i_k \le n} P(A_{i_1} \cap \cdots \cap A_{i_k})} = \sum_{1 \le i_1 < \cdots < i_k \le n} \frac{(n-k)!}{n!}$$

$$= \frac{(n-k)!}{n!} \cdot \#\{(i_1, \dots, i_k) : 1 \le i_1 < i_2 \cdots < i_k \le n\}$$

$$\cdot \#\{(i_1, \dots, i_k) : 1 \le i_1 < i_2 \cdots < i_k \le n\}$$

$= \#$ OF k-ELEMENT SUBSETS OF $\{1, \dots, n\}$

$$= \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\frac{\cancel{(n-k)!}}{\cancel{n!}} \times \frac{\cancel{n!}}{\cancel{(n-k)!}\; k!} = \boxed{\frac{1}{k!}}$$

$$P(A_1 \cup \dots \cup A_n) = \sum_{k=1}^{n} (-1)^{k+1} \left[\frac{1}{k!} \right]$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

$$P(E) = \underbrace{1}_{(-1)^0 \cdot \frac{1}{0!}} + \sum_{k=1}^{n} (-1)^{k} \frac{1}{k!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$\lim_{n \to \infty} P(E_n) = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$x = -1$

$$= e^{-1} = \frac{1}{e}$$

REMINDER : OFF HOURS AT 4 PM