MATH 201 (SUMMER 2023, SESH A2)

LECTURE 4 : 05/18/23

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LECTURES :
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. LECTURE 3 IS UPLOADED. (PANOPTO / SCHEDULE)
2. UPCOMING DEADLINES:
   - (a) HW 1 (FRI, MAY 19th AT 11 PM ET) → STRICT.
   - (b) WW 2 (SAT MAY 20th AT 11 PM ET) → LOOSE.
3. WW 1 IS CLOSED.
4. OFFICE HOURS TODAY 4-5 PM ET
5. COLLABORATION IN HW 1. → NAMES.
6. IMP: FILL OUT $\boxed{\text{FORM.}}$ → EXAM SCHEDULING. (LINK ON BLACKBOARD)
7. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!

NOTATION: IF A & B ARE EVENTS, WE WRITE AB FOR THE EVENT IN WHICH BOTH OCCUR.

FORMALLY,

$$AB = A \cap B$$

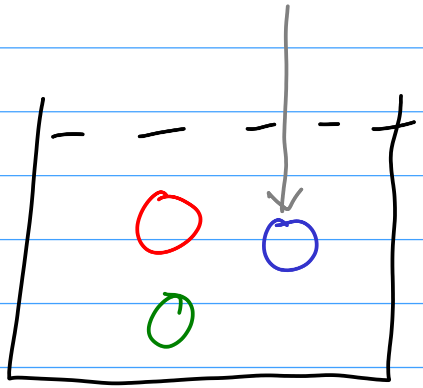
A AND B

$$P(AB) = P(A \cap B)$$

§ 2.1 CONDITIONAL PROBABILITY

RANDOMNESS CAN BE THOUGHT OF AS LACK OF INFORMATION

e.g. TEMPERATURE $\longrightarrow$ [30, 100]



DAY-TO-DAY VARIATION OF AT MOST 20°

e.g. $\Omega = \{1, 2, 3\}$

$P(1) = \frac{1}{5}$, $P(2) = \frac{2}{5}$, $P(3) = \frac{2}{5}$

PROMISE: THE RESULT OF THE EXPERIMENT IS EITHER 1 OR 2.

WHAT IS THE NEW PROBABILITY $\tilde{P}$?

$\tilde{P}(1) \propto P(1)$ & $\tilde{P}(2) \propto P(2)$ $\quad \tilde{P}(3) = 0$

$\tilde{P}\{1,2\} = \tilde{P}(1) + \tilde{P}(2) = 1$ $\quad \tilde{P}(2) = 2\tilde{P}(1) \Longrightarrow$ $\tilde{P}(1) = 1/3$, $\tilde{P}(2) = 2/3$, $\tilde{P}(3) = 0$

$$\frac{\tilde{P}(1)}{1 = \tilde{P}\{1,2\}} = \frac{P(1)}{P\{1,2\}} \Rightarrow \tilde{P}(1) = \frac{P(1)}{P\{1,2\}} = \frac{1/5}{1/5 + 2/5} = \frac{1}{3}$$

$$\frac{\tilde{P}(2)}{1 = \tilde{P}\{1,2\}} = \frac{P(2)}{P\{1,2\}} \Rightarrow \tilde{P}(2) = \frac{P(2)}{P\{1,2\}} = \frac{2}{3}$$

IMP!

Definition 2.1. Let B be an event in the sample space Ω such that $P(B) > 0$. Then for all events A the **conditional probability of A given B** is defined as

$$P(A \mid B) = \frac{P(AB)}{P(B)}. \quad = \frac{P(A \cap B)}{P(B)} \tag{2.2}$$

$\tilde{P}(B) = 1$

$\tilde{P}(\cdot) = P(\cdot \mid B)$

Fact 2.2. Let B be an event in the sample space Ω such that $P(B) > 0$. Then, as a function of the event A, the conditional probability $P(A \mid B)$ satisfies the axioms of Definition 1.1.

KOLMOGOROV'S.

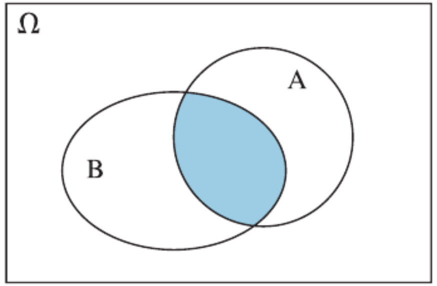


Figure 2.1. Venn diagram representation of conditioning on an event. The conditional probability of A given B is the probability of the part of A inside B (the shaded region), divided by the probability of B.

Example 2.3. Counting outcomes as in Example 1.7, the probability of getting 2 heads out of three coin flips is 3/8. Suppose the first coin flip is revealed to be heads. Heuristically, the probability of getting exactly two heads out of the three is now 1/2. This is because we are simply requiring the appearance of precisely one heads in the final two flips of the coin, which has a probability of 1/2.

O O O

A = FLIPPED EXACTLY TWO HEADS.

$$P(A) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{3}{2^3} = \frac{3}{8}$$

H H T
H T H
T H H

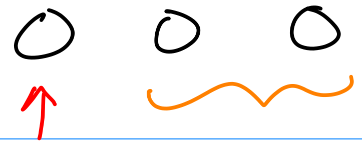
B = FIRST FLIP IS HEADS

O O

H H
T H
H T
T T

$P(A|B) = 1/2$ → INTUITIVE ANS

$P(B) = 1/2$



$$P(A \cap B) = \frac{\#\ \text{FAVOR}}{\#\ \text{TOTAL}} = \frac{2}{2^3} = \frac{1}{4}$$

$\begin{pmatrix} H & T & H \\ H & H & T \end{pmatrix}$

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

Fact 2.4. Suppose that we have an experiment with finitely many equally likely outcomes and B is not the empty set. Then, for any event A

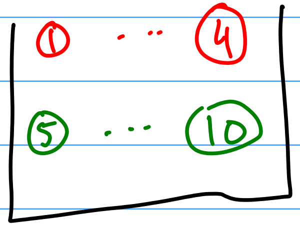
$$P(A|B) = \frac{\#AB}{\#B}.$$

← # FAVORABLE

← # TOTAL OUTCOMES GIVEN B

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\#A\cap B / \#\Omega}{\#B / \#\Omega} = \frac{\#AB}{\#B}$$

Example 2.5. We have an urn with 4 red and 6 green balls. We choose a sample of 3 without replacement. Find the conditional probability of having exactly 2 red balls in the sample given that there is at least one red ball in the sample.



ORDER?

$\downarrow$

EITHER IS FINE

A = EXACTLY 2 RED BALLS

B = $\geq$ 1 RED BALL

$$\frac{P(A \cap B)}{P(B)}$$

MODEL: ORDER DOESN'T MATTER

B^c = NO RED BALL $\Rightarrow P(B^c) = \dfrac{\binom{6}{3}}{\binom{10}{3}} = \dfrac{1}{6}$

$$P(B) = 1 - P(B^c) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$A \cap B = A \qquad (\text{BECAUSE } A \subseteq B)$$

$$P(AB) = P(A) = \frac{\binom{4}{2}\binom{10}{1}}{\binom{10}{3}} = \frac{3}{10}$$

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{3/10}{5/6} = \frac{9}{25}$$

$$P(A \mid B) = \frac{P(AB)}{\boxed{P(B)}} \quad > 0$$

MULTIPLICATION RULE

$$P(AB) = P(B)\,P(A \mid B)$$

SOMETIMES B & $A|B$ ARE EASIER TO UNDERSTAND THAN AB.

Example 2.7. Suppose an urn contains 8 red and 4 white balls. Draw two balls without replacement. What is the probability that both are red?

ORDER MATTERS

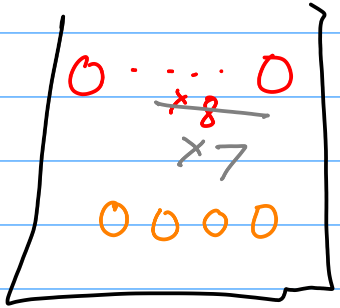
R_1 = 1st BALL DRAWN IS RED

R_2 = 2nd BALL DRAWN IS RED

$$P(R_1R_2) = P(R_1)\cdot P(R_2 \mid R_1)$$

MULT. RULE
[$A = R_2$, $B = R_1$]

$$P(R_1) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{8}{12} = \frac{2}{3}$$



$$P(R_2 \mid R_1) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{7}{11}$$

$$P(R_1 R_2) = P(R_1)\, P(R_2 \mid R_1)$$

$$= \frac{2}{3} \cdot \frac{7}{11} = \frac{14}{33}$$

Fact 2.6. (Multiplication rule for n events) If $A_1, \ldots, A_n$ are events and all the conditional probabilities below make sense then we have

$$P(A_1A_2\cdots A_n) = \underline{\underline{P(A_1)}}\,\underline{\underline{P(A_2 \mid A_1)}}\,\underline{\underline{P(A_3 \mid A_1A_2)}}\cdots P(A_n \mid A_1\cdots A_{n-1}). \qquad (2.5)$$

Pf. INDUCTION.

BASE CASE $(n=1)$ $\quad P(A_1) = P(A_1)$

$(n=2) \to \quad P(A_1A_2) = P(A_1)P(A_2 \mid A_1)$

$n=3 \quad P(A_1\underbrace{A_2A_3}_{B}) = P(A_1)\,P(B \mid A_1)$

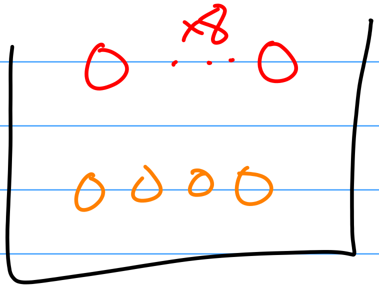
$= P(A_1)\,P(A_2A_3 \mid A_1) = P(A_1)\,P(A_2 \mid A_1)\,P(A_3 \mid A_1A_2)$

$P(A_2A_3 \mid A_1) = P(A_2 \mid A_1)\cdot P(A_3 \mid A_1A_2)$

ORDER MATTERS

Example Suppose an urn contains 8 red and 4 white balls. Draw ~~two~~ 4 balls

$P\{\text{first two draws are red and the third and fourth draws are white}\}$



$R_j \to$ j th DRAW IS RED

$W_j \to$ j th DRAW IS WHITE

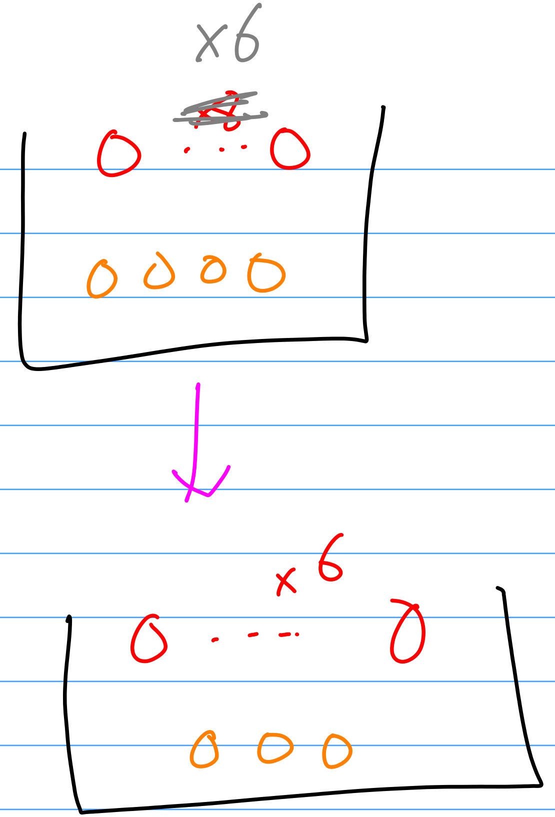
$$P(R_1 R_2 W_3 W_4) = P(R_1) \cdot P(R_2 \mid R_1) \cdot P(W_3 \mid R_1 R_2) \cdot P(W_4 \mid R_1 R_2 W_3)$$

$$P(R_1) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{8}{12} = \frac{2}{3}$$

$$P(R_2 \mid R_1) = 7/11$$

$$P(W_3 \mid R_1 R_2) = \frac{\#\ \text{FAVOR}}{\#\ \text{TOTAL}}$$

$$= \frac{4}{10} = \frac{2}{5}$$



$$P(W_4 \mid R_1 R_2 W_3) = \frac{3}{9} = \frac{1}{3}$$

$$P(R_1 R_2 W_3 W_4) = P(R_1) \cdot P(R_2 \mid R_1) \cdot P(W_3 \mid R_1 R_2) \cdot P(W_4 \mid R_1 R_2 W_3)$$

$$= \frac{2}{3} \cdot \frac{7}{11} \cdot \frac{2}{5} \cdot \frac{1}{3} = \frac{28}{495}$$

$\Omega = \{(c,b)\}$

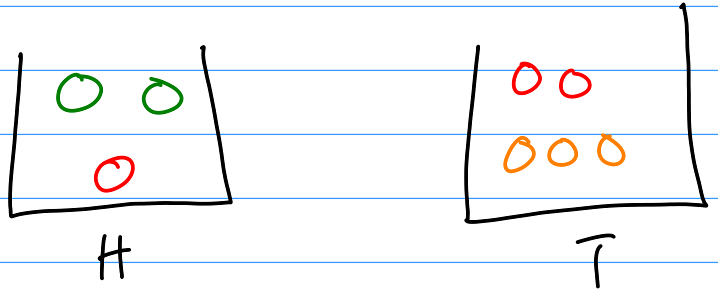
NOT E.L.O.

Example 2.8. We have two urns. Urn I has 2 green balls and 1 red ball. Urn II has 2 red balls and 3 yellow balls. We perform a *two-stage experiment*. First choose one of the urns with $\boxed{\text{equal probability}}$. Then sample one ball uniformly at random from the selected urn.

BY COIN FLIP

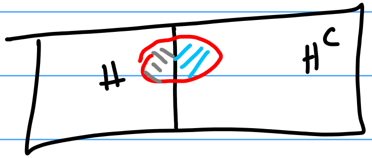
What is the probability that we draw a red ball?

What is the probability that we draw a green ball?



H = COIN FLIP IS HEADS

H^C = COIN FLIP IS TAILS



R = BALL IS RED

G = BALL IS GREEN

WANT THIS.

$$\boxed{P(R)} = P(R \cap H^C) + P(R \cap H)$$

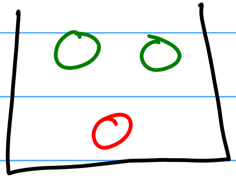
$$P(G) = P(G \cap H^C) + P(G \cap H)$$

$$P(RH) = P(H)\,P(R \mid H)$$

$$P(RH^c) = P(H^c)\,P(R \mid H^c)$$

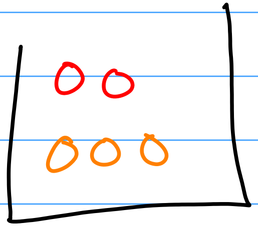
$$P(R) = P(H)\,P(R \mid H) + P(H^c)\,P(R \mid H^c)$$

$$= \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{2}{5} = \frac{1}{6} + \frac{1}{5} = \frac{11}{30}$$



$\frac{1}{3}$

H



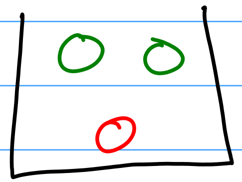
$\frac{2}{5}$

T

$$P(G) = P(GH) + P(GH^c)$$

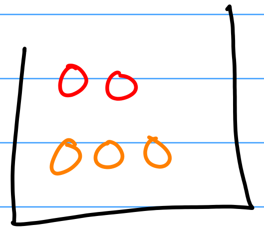
$$= P(H)\,P(G \mid H) + P(H^c)\,P(G \mid H^c)$$

$$= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times 0 = \frac{1}{3}$$



H

$\frac{2}{3}$



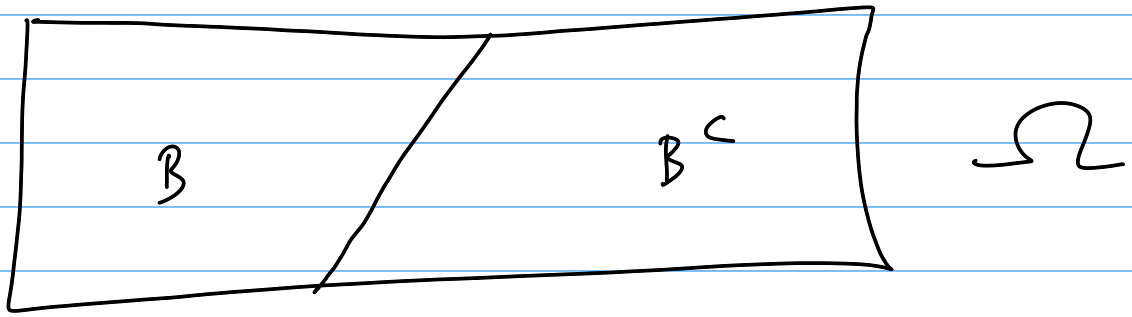
T

0

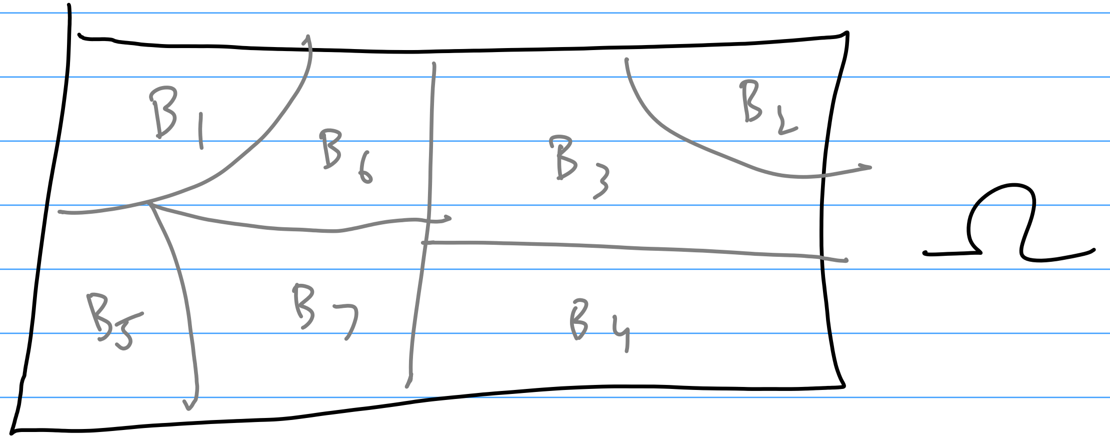
IDEA :

$$P(A) = P(AB) + P(AB^c)$$
$$= P(A|B)\,P(B) + P(A|B^c)\,P(B^c)$$

WHAT ARE WE USING ABOUT B & B^C ?



Definition 2.9. A finite collection of events $\{B_1, \ldots, B_n\}$ is a **partition** of Ω if the sets B_i are pairwise disjoint and together they make up Ω. That is, $B_iB_j = \varnothing$ whenever $i \neq j$ and $\bigcup_{i=1}^{n} B_i = \Omega$.



$\cup B_i = \Omega \quad , \quad B_iB_j = \phi$

$$A \cap \left[\cup B_j \right] = A \cap \Omega = A$$

$$A \cap \left[\cup B_j \right] = \bigcup_j \left[\underline{\underline{A \cap B_j}} \right]$$

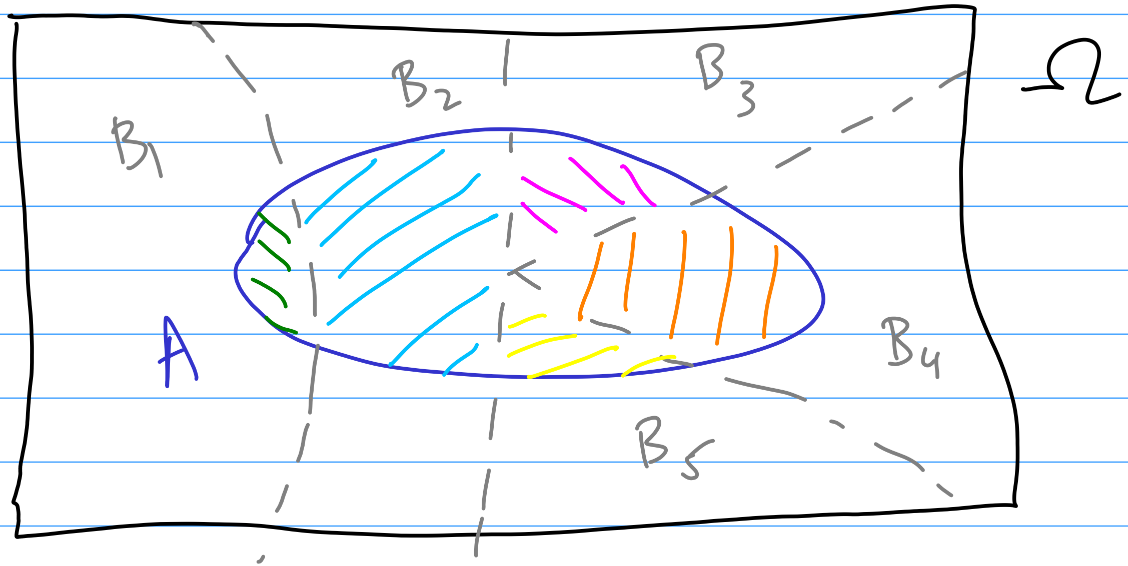
$$P(A) = P\left(\bigcup_j \left[A \cap B_j \right] \right) = \sum_j P(AB_j)$$

$$P(A) = \sum_j P(B_j)\, P(A \mid B_j)$$

Fact 2.10. Suppose that $B_1, \dots, B_n$ is a partition of Ω with $P(B_i) > 0$ for $i = 1, \dots, n$. Then for any event A we have

$$P(A) = \sum_{i=1}^{n} P(AB_i) = \sum_{i=1}^{n} P(A|B_i)P(B_i). \tag{2.7}$$

LAW OF TOTAL PROBABILITY



Example 2.11. There are three types of coins in circulation. 90% of coins are fair coins that give heads and tails with equal probability. 9% of coins are moderately biased and give tails with probability $\boxed{3/5}$. The remaining 1% of coins are heavily biased and give tails with probability $\boxed{9/10}$. The type of a coin cannot be determined from its appearance. I have a randomly chosen coin in my pocket and I flip it. What is the probability I get tails?

A = FLIP IS TAILS

B_1 = COIN IS FAIR $\qquad P(B_1) = \frac{90}{100} = \frac{9}{10}$

B_2 = COIN IS MOD. BIASED $\qquad P(B_2) = \frac{9}{100}$

B_3 = COIN IS HEAVILY BIASED $\qquad P(B_3) = \frac{1}{100}$

$B_j \rightarrow$ ARE PAIR-WISE

$\{B_j\}$ PARTITION Ω

$$\begin{cases} B_1 B_2 = B_2 B_3 = B_3 B_1 = \phi \\ \bigcup_{j=1}^{3} B_j = \Omega \end{cases}$$

$$P(A) = \sum_{j=1}^{3} P(B_j) \; P(A \mid B_j)$$

$$= \underline{\underline{P(B_1)}} \; \underline{\underline{P(A \mid B_1)}} + \underline{\underline{P(B_2)}} \; \underline{\underline{P(A \mid B_2)}} + \underline{\underline{P(B_3)}} \; \underline{\underline{P(A \mid B_3)}}$$

$$P(A) = P(B_1)\,P(A \mid B_1) + P(B_2)\,P(A \mid B_2) + P(B_3)\,P(A \mid B_3)$$

$$= \frac{9}{10} \cdot \frac{1}{2} + \frac{9}{100} \cdot \frac{3}{5} + \frac{1}{100} \cdot \frac{9}{10} = \frac{450}{1000} + \frac{54}{1000} + \frac{9}{1000}$$

$$= \frac{513}{1000} = 51.3\%$$

$$P(A \mid B_1) = P\left(\begin{array}{c}\text{YOU FLIP TAILS} \\ \text{GIVEN THE COIN} \\ \text{IS FAIR}\end{array}\right) = \frac{1}{2}$$

$$P(A \mid B_2) = P\left(\begin{array}{c}\text{FLIPPING TAILS} \\ \text{GIVEN MOD. BIASED}\end{array}\right) = \frac{3}{5}$$

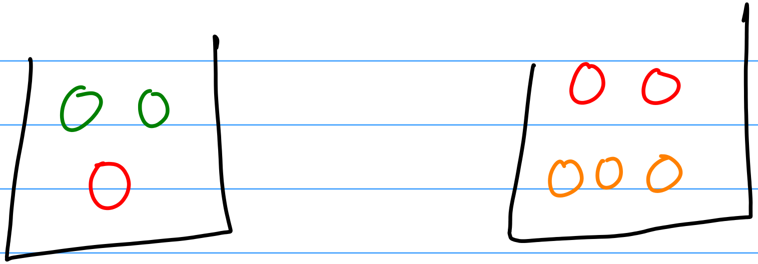
$$P(A \mid B_3) = P\left(\text{TAILS} \;\middle|\; \begin{array}{c}\text{HEAVILY} \\ \text{BIASED}\end{array}\right) = \frac{9}{10}$$

BACK AT

10 :30 AM

§ 2.2 BAYES' FORMULA

Example 2.12. As in Example 2.8, we have two urns: urn I has 2 green balls and 1 red ball, while urn II has 2 red balls and 3 yellow balls. An urn is picked randomly and a ball is drawn from it. Given that the chosen ball is red, what is the probability that the ball came from urn I?



H T

G = BALL IS GREEN

R = BALL IS RED

H = HEADS IN FLIP (PICKED URN 1)

H^c = TAILS IN FLIP (PICKED URN 2)

$P(H \mid R)$

$$P(H|R) = \frac{P(HR)}{P(R)} = \frac{P(HR)}{P(HR)+P(H^cR)} = \boxed{\frac{P(H)\,P(R|H)}{P(H)P(R|H)+P(H^c)\,P(R|H^c)}}$$

BAYES' FORMULA.

H & H^c PARTITION

$$P(R) = P(H)P(R|H) + P(H^c)P(R|H^c)$$

$P(H) = P(H^c) = 1/2$

$P(R|H) = \frac{1}{3}$

$P(R|H^c) = \frac{2}{5}$

$$P(H|R) = \frac{\frac{1}{2}\times\frac{1}{3}}{\frac{1}{2}\times\frac{1}{3}+\frac{1}{2}\times\frac{2}{5}}$$

$$= \frac{1/6}{1/6+1/5} = \frac{5}{11}$$

Fact 2.13. (Bayes' formula) If $P(A), P(B), P(B^c) > 0$, then

$$P(B \mid A) = \frac{P(AB)}{P(A)} = \frac{P(A \mid B)\, P(B)}{P(A \mid B)\, P(B) + P(A \mid B^c)\, P(B^c)}. \tag{2.9}$$

<u>BAYESIAN THINKING</u>: PROBABILITY AS A STUDY OF UNCERTAINTY.

$P(B) \rightarrow$ PRIOR PROBABILITY

$P(B|A) \rightarrow$ POSTERIOR PROBABILITY

Example 2.14. Suppose we have a medical test that detects a particular disease 96% of the time, but gives false positives 2% of the time. Assume that $\boxed{0.5\%}$ of the population carries the disease. If a random person tests positive for the disease, what is the probability that they actually carry the disease?

D = PERSON TESTED HAS DISEASE, D & D^c PARTITION Ω

$+$ = PERSON TESTS POSITIVE

$$P(D \mid +) = \frac{P(D \cap +)}{P(+)} = \frac{P(D)\,P(+ \mid D)}{P(D)\,P(+ \mid D) + P(D^c)\,P(+ \mid D^c)}$$

$P(D) = \frac{0.5}{100} = \frac{1}{200}$, $P(D^c) = \frac{199}{200}$

$P(+ \mid D) = 96/100 = \frac{24}{25}$

$P(+ \mid D^c) = 2/100 = \frac{1}{50}$

$$P(D \mid +) = \frac{P(D)\,P(+ \mid D)}{P(D)\,P(+ \mid D) + P(D^c)\,P(+ \mid D^c)}$$

$$= \frac{\frac{1}{200} \times \frac{24}{25}}{\frac{1}{200} \times \frac{24}{25} + \frac{199}{200} \times \frac{1}{50}} \approx 0.194$$

WHY DO DOCTORS TRUST MEDICAL TESTS?

ANS: WHAT IF P(DISEASE) IS HIGHER (= 1/2)

$$P(D) = \frac{1}{2} \qquad P(D^c) = \frac{1}{2}$$

$$P(D \mid +) = \frac{P(D)\,P(+ \mid D)}{P(D)\,P(+ \mid D) + P(D^c)\,P(+ \mid D^c)} = \frac{\frac{1}{2} \cdot \frac{96}{100}}{\frac{1}{2} \cdot \frac{96}{100} + \frac{1}{2} \cdot \frac{4}{100}}$$

$$\simeq 0.98$$

$P(A|B)$ vs. $P(B|A)$

Fact 2.15. (General version of Bayes' formula) Let $B_1, \ldots, B_n$ be a partition of the sample space Ω such that each $P(B_i) > 0$. Then for any event A with $P(A) > 0$, and any $k = 1, \ldots, n$,

$$P(B_k \mid A) = \frac{P(AB_k)}{P(A)} = \frac{P(A \mid B_k)\, P(B_k)}{\sum_{i=1}^{n} P(A \mid B_i)\, P(B_i)}. \tag{2.10}$$

PRIOR & POSTERIOR.

$P(B_k) \to$ PRIOR

$P(B_k \mid A) \to$ POSTERIOR

$\to$ ELECTION FORECASTING.

100 $\to$

50	(1)
10	(2)
40	(3)

Example 2.16. Return to Example 2.11 with three types of coins: fair (F), moderately biased (M) and heavily biased (H), with probabilities of tails

$$P(\text{tails} \mid F) = \frac{1}{2}, \quad P(\text{tails} \mid M) = \frac{3}{5}, \quad \text{and} \quad P(\text{tails} \mid H) = \frac{9}{10}.$$

We hold a coin of unknown type. The probabilities of its type were given by

$$P(F) = \frac{90}{100}, \quad P(M) = \frac{9}{100}, \quad \text{and} \quad P(H) = \frac{1}{100}.$$

These are the prior probabilities. We flip the coin once and observe tails. Bayes' formula calculates our new posterior probabilities.

$P(T) = \frac{513}{1000}$

T = FLIP TURNS OUT TAILS.

$$P(F \mid T) = \frac{P(F)\,P(T \mid F)}{P(F)\,P(T \mid F) + P(M)\,P(T \mid M) + P(H)\,P(T \mid H)} ?$$

$$= \frac{\frac{9}{10} \cdot \frac{1}{2}}{513/100} = \frac{45}{513} \approx 0.877$$

$$P(M \mid T) = \frac{P(M)\,P(T \mid M)}{P(T)} = \frac{\frac{9}{100} \cdot \frac{3}{5}}{513/1000} = \frac{54}{513}$$

$$\simeq 0.105$$

$$P(H \mid T) = \frac{P(H)\,P(T \mid H)}{P(T)} = \frac{1/100 \cdot \frac{9}{10}}{513/1000} = \frac{9}{513}$$

$$\simeq 0.018$$

PRIOR	POSTERIOR	
$P(F) = \frac{90}{100} = 0.9$	$P(F \mid T) \simeq 0.877$	$\downarrow$
$P(M) = \frac{9}{100} = 0.09$	$P(M \mid T) \simeq 0.105$	$\uparrow$
$P(H) = 1/100 = 0.01$	$P(H \mid T) \simeq 0.018$	$\uparrow$

REMINDER : OFF HRS AT 4PM TODAY