MATH 201 (SUMMER 2023, SESH A2)

LECTURE 5 : 05/22/23

ANURAG SAHAY
OFF HRS: BY APPT (VIA ZOOM)

email: anuragsahay@rochester.edu

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

Zoom ID:
979-4693-0650

COURSE WEB PAGE https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. LECTURE 4, WEEK 2 H.W. IS UPLOADED. (PANOPTO/WEBSITE)

2. OFFICE HOURS: TR, 11:15 AM – 12:15 PM ET.
(TUES/THURS
RIGHT AFTER
CLASS)

3. DEADLINES:
   a. WW03 – TUES, MAY 23rd
   b. HW02 – TUES, MAY 23rd
   c. WW04 – FRI, MAY 26th
   d. HW03 – SAT, MAY 27th → TO BE UPLOADED

   AT 11 PM.

4. WRITTEN HOMEWORK → PLEASE EXPLAIN YOUR WORK.
   → 1/3 FOR ANSWER
   → 2/3 FOR EXPL.

5. NO IN-CLASS LECTURE ON WEDNESDAY (MAY 24th) → WILL BE RECORDED.

6. PLEASE KEEP YOUR VIDEOS ON, IF POSSIBLE!

2.4 INDEPENDENCE

INDEPENDENT EVENTS

e.g. (1) SUCCESSIVE COIN-FLIPS OF A FAIR COIN
ARE HEADS (H) (H)

e.g. TODAY'S TEMPERATURE IS HIGHER THAN Y'DAY & A DIE ROLLS TO 1.

NON-e.g. THE PRICE OF GOOGLE & FACEBOOK STOCK BOTH FELL

ALPHABET META

DISRUPTION IN S_i REG. CHANGES

A, B INDEPENDENT?

HEURISTIC : $\boxed{P(A|B) = P(A)}$

$(P(B) > 0)$

L.H.S. $= \dfrac{P(A \cap B)}{P(B)} = P(A)$

$\Rightarrow \boxed{P(A \cap B) = P(A)\,P(B)}$

DEFN.

$P(B|A) = P(B)$

Definition 2.17. Two events A and B are **independent** if

$$P(AB) = P(A)P(B). \tag{2.11}$$

$AB = A \cap B$

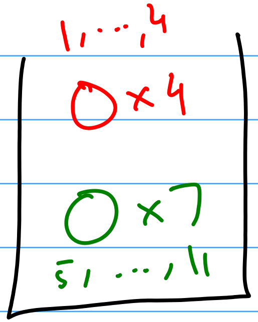
N.B.: THIS ALLOWS $P(A)$ OR $P(B)$ TO BE 0.

SYMMETRIC IN A & B

Example 2.19. Suppose that we have an urn with 4 red and 7 green balls. We choose two balls with replacement. Let

$$A = \{\text{first ball is red}\} \quad \text{and} \quad B = \{\text{second ball is green}\}.$$

Is it true that A and B are independent? What if we sample without replacement?



$$P(AB) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{4\cdot 7}{11^2} = \frac{4}{11}\cdot\frac{7}{11}$$

$$P(B) = \frac{11\cdot 7}{11^2} = \frac{7}{11}, \quad P(A) = \frac{4\cdot 11}{11^2} = \frac{4}{11}$$

$$P(AB) = P(A)P(B)$$

HEURISTIC $\rightarrow$ A OCCURS $\Rightarrow$ B IS MORE LIKELY

$Q \rightsquigarrow$ PROB.

$$Q(AB) = \frac{\#\text{FAVOR}}{\#\text{TOTAL}} = \frac{4 \cdot 11}{11 \cdot 10} = \frac{2}{5}$$

$$Q(A) = \frac{4 \cdot 10}{11 \cdot 10} = \frac{4}{11}, \quad Q(B) = \frac{11 \cdot 7}{11 \cdot 10} = \frac{7}{10}$$

$$Q(A)Q(B) = \frac{4}{11} \cdot \frac{7}{10} \neq \frac{2}{5} = Q(AB)$$

Fact 2.20. Suppose that A and B are independent. Then the same is true for each of these pairs: A^c and B, A and B^c, and A^c and B^c.

Pf

A & B IND. $\Rightarrow$ A^c & B IND.

$$P(A^c B) = P(B) - \underbrace{P(AB)}_{P(A)\,P(B)}$$

[RECALL $P(B) = P(AB) + P(A^c B)$]

$$= P(B) - P(A)\,P(B)$$

$$= P(B)\underbrace{(1 - P(A))}_{P(A^c)} = P(A^c)\,P(B)$$

OTHERS ARE SIMILAR.

Example 2.21. Suppose that A and B are independent and $P(A) = 1/3$, $P(B) = 1/4$. Find the probability that exactly one of the two events is true.

$$E = \underbrace{A^c B}_{\text{B HAS OCC., A HASN'T}} \cup \underbrace{A B^c}_{\text{A HAS OCC., BUT B HASN'T}} \Rightarrow P(E) = \underbrace{P(A^c B)}_{P(A^c)\,P(B)} + \underbrace{P(AB^c)}_{P(A)\,P(B^c)}$$

BY IND.

$$P(E) = P(A^c)\,P(B) + P(A)\,P(B^c)$$

$$= \left(1 - \frac{1}{3}\right) \cdot \frac{1}{4} + \frac{1}{3} \cdot \left(1 - \frac{1}{4}\right)$$

$$= \frac{2}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{3}{4} = \frac{2+3}{12} = \frac{5}{12}$$

WHAT IF WE HAVE >2 EVENTS ?

e.g. $A_1, A_2, \dots, A_n$

$$\boxed{P(A_1 A_2 \dots A_n) = P(A_1) P(A_2) \dots P(A_n)}$$

REASONABLE DEFN ?

NOT GOOD ENOUGH

$P([a,b])$
$= b - a$

Example 2.24. Choose a random real number uniformly from the unit interval $\Omega = [0, 1]$. (This model was introduced in Example 1.17.) Consider these events:

$$\boxed{A = [\tfrac{1}{2}, 1]}, \quad \boxed{B = [\tfrac{1}{2}, \tfrac{3}{4}]}, \quad \boxed{C = [\tfrac{1}{16}, \tfrac{9}{16}]}.$$

Then $\boxed{ABC} = [\frac{1}{2}, \frac{9}{16}]$ and

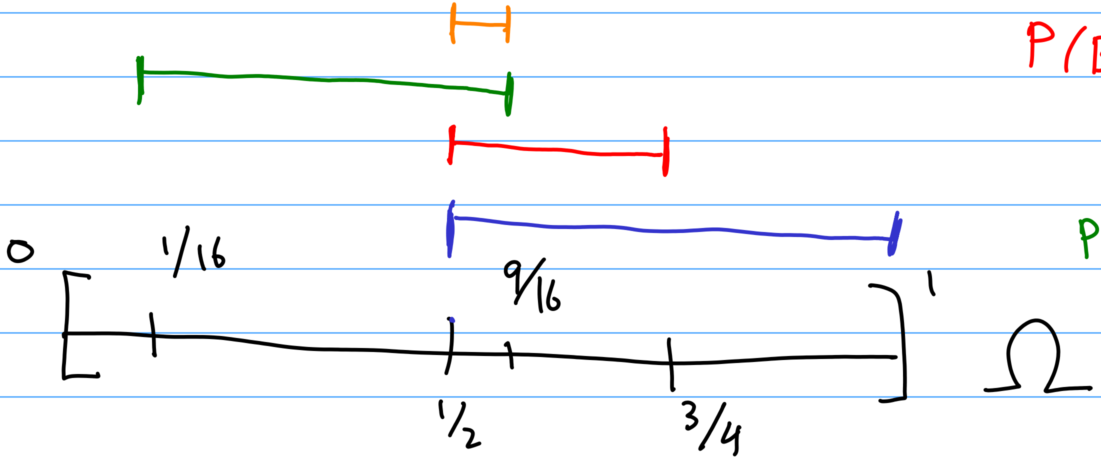
$$P(ABC) = \boxed{\frac{1}{16}} = \boxed{\frac{1}{2}} \cdot \boxed{\frac{1}{4}} \cdot \boxed{\frac{1}{2}} = P(A)P(B)P(C).$$

$P(ABC) = \frac{9}{16} - \frac{1}{2} = \frac{1}{16}$

$P(B) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(C) = \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$



SHOULD THESE BE THOUGHT OF AS INDEPENDENT EVENTS?

$B = [1/2, 3/4]$

$A = [1/2, 1]$

$B \subseteq A \longrightarrow$ CANNOT BE INDEPENDENT!

Definition 2.22. Events $A_1, \ldots, A_n$ are independent (or mutually independent) if for every collection $A_{i_1}, \ldots, A_{i_k}$, where $2 \le k \le n$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$,

$$P(A_{i_1} A_{i_2} \cdots A_{i_k}) = P(A_{i_1}) P(A_{i_2}) \cdots P(A_{i_k}). \tag{2.12}$$

e.g. $n=3$ $\quad A_1, A_2, A_3$

$\{A_1, A_2, A_3\} \to P(A_1 A_2 A_3) = P(A_1)P(A_2)P(A_3)$

$\{A_1, A_2\} \to P(A_1 A_2) = P(A_1)P(A_2)$

$\{A_1, A_3\} \to P(A_1 A_3) = P(A_1)P(A_3)$

$\{A_2, A_3\} = P(A_2 A_3) = P(A_2)P(A_3)$

e.g. $n=4$

$P(A_1 A_2 A_3 A_4) = \prod_{j=1}^{4} P(A_j)$

$P(A_1 A_3) = P(A_1)P(A_3)$

$P(A_1 A_3 A_4) = P(A_1)P(A_3)P(A_4)$

4

Fact 2.23. Suppose events $A_1, \ldots, A_n$ are mutually independent. Then for every collection $A_{i_1}, \ldots, A_{i_k}$, where $2 \le k \le n$ and $1 \le i_1 < i_2 < \cdots < i_k \le n$, we have

$$P(A^*_{i_1} A^*_{i_2} \cdots A^*_{i_k}) = P(A^*_{i_1}) P(A^*_{i_2}) \cdots P(A^*_{i_k}) \tag{2.13}$$

where each A^*_i represents either $\underline{A_i}$ or $\underline{A^c_i}$.

→ NOT HARD BUT TEDIOUS.

A, B IND. $\Rightarrow$ A^c, B IND.

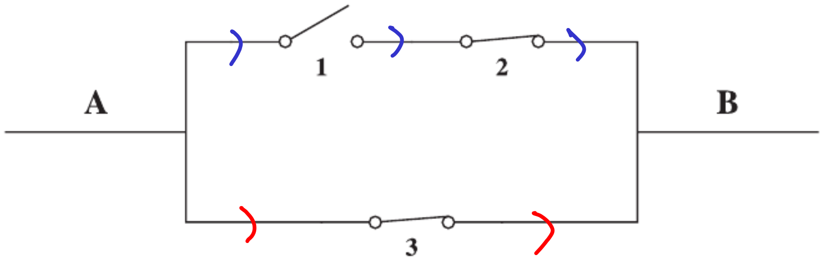
e.g.

$$P(A_1 A_5^c A_6 A_7^c) = P(A_1) P(A_5^c) P(A_6) P(A_7^c)$$

$$P(A_2 A_3^c) = P(A_2) P(A_3^c)$$

Example 2.26. The picture below represents an electric network. Current can flow through the top branch if switches 1 and 2 are closed, and through the lower branch if switch 3 is closed. Current can flow from A to B if current can flow either through the top branch or through the lower branch.

Assume that the switches are open or closed independently of each other, and that switch i is closed with probability p_i. Find the probability that current can flow from point A to point B.



S_j = SWITCH j IS CLOSED

E = CURRENT FLOWS

$= \underbrace{(S_1 S_2) \cup S_3}_{\text{NOT DISJOINT}}$

e.g. FLIP 3 FAIR COINS.

$$P(E) = \overbrace{P(S_1 S_2)}^{P(S_1)P(S_2)} + P(S_3) - \underbrace{P(S_1 S_2 \cap S_3)}_{P(S_1)\,P(S_2)\,P(S_3)}$$

$$\{S_1, S_2, S_3\} \to \text{IND}$$

$$p_j = P(S_j)$$

$$P(E) = P(S_1)\,P(S_2) + P(S_3) - P(S_1)\,P(S_2)\,P(S_3)$$

$$= p_1 p_2 + p_3 - p_1 p_2 p_3$$

$$p_1 = p_2 = p_3 = 1/2$$

$$P(E) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{2+4-1}{8} = \frac{5}{8}$$

$\S$ 1.5 RANDOM VARIABLES: A FIRST LOOK

INTUITIVE DEFN: A RANDOM VARIABLE IS A REAL NUMBER WHOSE VALUE DEPENDS ON A RANDOM PROCESS

e.g.: ROLLING TWO DICE $\rightarrow$ SUM

THE FACE VALUE OF A CARD   1, ..., 10
J − 11, Q − 12
K − 13

PRICE OF A STOCK

FORMAL DEFN:

> **Definition 1.28.** Let Ω be a sample space. A **random variable** is a function from Ω into the real numbers. ♣ $X : \Omega \longrightarrow \mathbb{R}$

NOTE:

1. A RANDOM VARIABLE IS FORMALLY A FUNCTION NOT A VARIABLE.
2. HOWEVER, KEEP THE INTUITION OF A REAL NUMBER AT THE BACK OF YOUR MIND.
3. USUALLY DENOTED BY $\underbrace{X, Y, Z}$ NOT f, g, h

Example 1.29. We consider again the roll of a pair of dice (Example 1.6). Let us introduce three random variables: X_1 is the outcome of the first die, X_2 is the outcome of the second die, and S is the sum of the two dice. [illegible]

$$\Omega = \{(i, j) : i, j \in \{1, \dots, 6\}\}$$

$$\omega = (5, 1) \rightsquigarrow X_1 = 5, \quad X_2 = 1 \qquad S = 5 + 1 = 6$$

$$X_1(5,1) = 5 \qquad X_2(5,1) = 1 \qquad S(5,1) = 6$$

$$X_1(i,j) = i \qquad X_1, X_2 \in \{1, \dots, 6\}$$

$$X_2(i,j) = j \qquad S(i,j) = i + j \in \{2, \dots, 12\}$$

$\to$ DISC.

USING RANDOM VARIABLES, WE CAN DEFINE EVENTS

$$\Omega = \{1,\dots,6\}^2$$

e.g. $\{S = 8\} = \{(i,j) \in \Omega : S(i,j) = 8\}$

$$= \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$$

AND, INTERSECTION

$$\{X_1 = 1, X_2 = 5\} = \{\omega \in \Omega : X_1(\omega) = 1, X_2(\omega) = 5\}$$

$$= \{(1,5)\}$$

$(= \{X_1 = 1\} \cap \{X_2 = 5\})$

$$\{X_1 = 5 \ \boxed{\text{OR}}\ X_2 = 1\} = \{X_1 = 5\} \cup \{X_2 = 1\}$$

UNION

$$\Omega = \{\underbrace{1, 2, 3}_{-\$1}, \underbrace{4}_{+\$1}, \underbrace{5, 6}_{+\$3}\}$$

Example 1.30. A die is rolled. If the outcome of the roll is 1, 2, or 3, the player loses \$1. If the outcome is 4, the player gains \$1, and if the outcome is 5 or 6, the player gains \$3. Let W denote the change in wealth of the player in one round of this game.

DISCRETE

$$W \in \{-1, 1, 3\}$$

$$\{W = -1\} = \{1, 2, 3\} \qquad P(W = -1) = 3/6 = 1/2$$

$$\{W = +1\} = \{4\} \qquad P(W = 1) = 1/6$$

$$\{W = +3\} = \{5, 6\} \qquad P(W = 3) = 2/6 = 1/3$$

$P(\omega \in [a,b]) = b - a$

e.g.: SELECT A POINT UNIFORMLY AT RANDOM FROM $[0,1]$ & LET Y BE TWICE THAT POINT.

FOR $a \in \mathbb{R}$, WHAT IS $P(Y \leq a)$?

$$P(Y \leq a) = P(\omega \in [0, a/2])$$

$$= \begin{cases} 0 & \text{IF } a < 0 \\ a/2 & \text{IF } 0 \leq a \leq 2 \\ 1 & \text{IF } a > 2 \end{cases}$$

NOT DISCRETE

$Y \in (0, \infty)$

Example 1.32. A random variable X is *degenerate* if there is some real value b such that $P(X = b) = 1$. ▲

ESSENTIALLY CONSTANT

NOTE: $\{X \neq b\}$ CAN BE NON-EMPTY,

BUT $P(X \neq b) = 0$.

RETURN AT

10 : 10 AM

Definition 1.33. Let X be a random variable. The **probability distribution** of the random variable X is the collection of probabilities $\boxed{P\{X \in B\}}$ for sets B of real numbers. ♣

$$\underbrace{\{X \in B\}}_{\text{MOST GENERAL QUESTION ONE CAN ASK}} = \{\omega \in \Omega : X(\omega) \in B\}$$

EXTREMELY COMPLICATED.

NOTE: FROM THIS POINT ON, WE WILL SUPPRESS Ω UNLESS ABSOLUTELY NECESSARY.

DISCRETE RANDOM VARIABLES

Definition 1.34. A random variable X is a **discrete random variable** if there exists a finite or countably infinite set $\{k_1, k_2, k_3, \ldots\}$ of real numbers such that

$$\sum_i P(X = k_i) = 1 \tag{1.27}$$

where the sum ranges over the entire set of points $\{k_1, k_2, k_3, \ldots\}$.

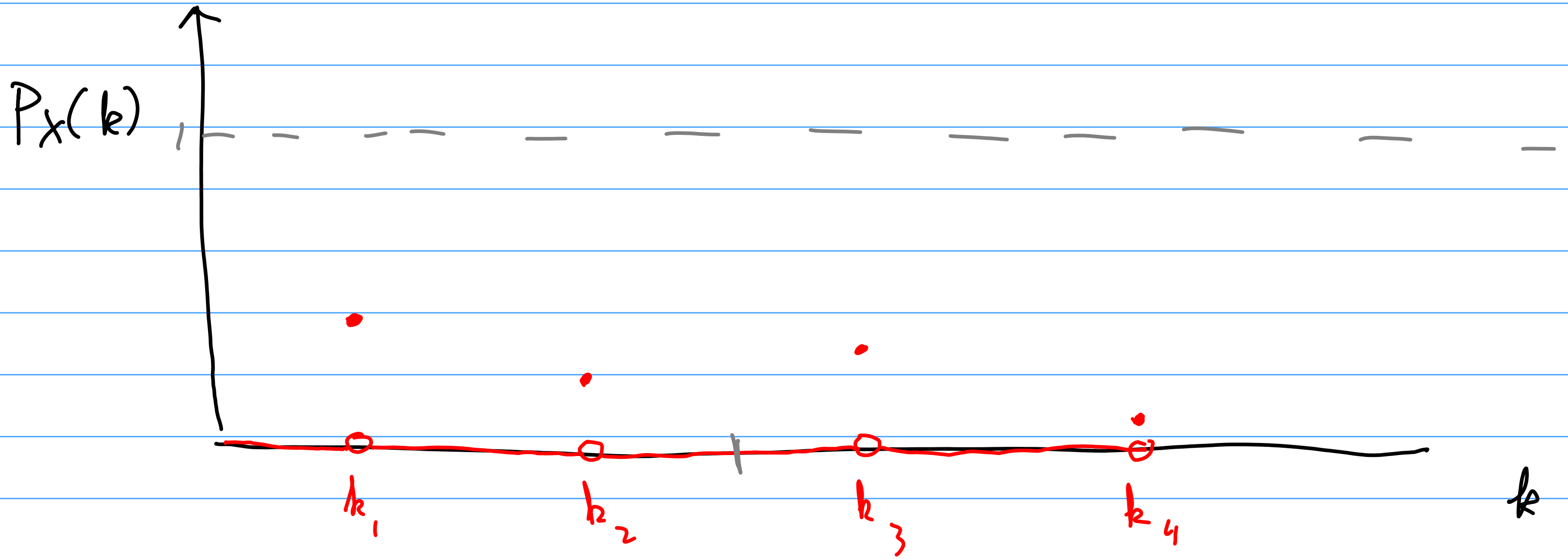
$X \in \{k_1, k_2, k_3, \ldots\}$ WITH $P = 1$.

LET'S GO OVER PREVIOUS EXAMPLES

Definition 1.35. The **probability mass function** (p.m.f.) of a discrete random variable X is the function p (or p_X) defined by

$$p(k) = P(X = k)$$

for possible values k of X.



P.M.F. $\longrightarrow$ COMPLETELY DETERMINES DISTRIBUTION.

$$P(X \in B) = P(X \in B \cap \{k_1, k_2, k_3, \dots\})$$

$$= P(X \in \{k : k \in B\})$$

↑ VALUES THAT X TAKES

(A.P.D.)

$$= \sum_{k:\, k \in B} \underbrace{P(X = k)}_{\text{p.m.f.}} = \sum_{k \in B} p_X(k)$$

↑ COUNTABLE OR FINITE.

IN PARTICULAR, $\sum_k p_X(k) = P(X \in \mathbb{R}) = 1.$

Example 1.36. (Continuation of Example 1.29) Here are the probability mass functions of the first die and the sum of the dice.

$X_1, X_2, S.$

$\Omega = \{1, \dots, 6\}^2$

k	1	2	3	4	5	6
$p_{X_1}(k) = P(X_1 = k)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

(1,2) OR (2,1)

p.m.f.

k	2	3	4	5	6	7	8	9	10	11	12
$p_S(k) = P(S = k)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$p_S(3) = P(S=3)$

$= \frac{2}{36}$

Probabilities of events are obtained by summing values of the probability mass function. For example,

$P(2 \le S \le 5) =$ [redacted] ▲

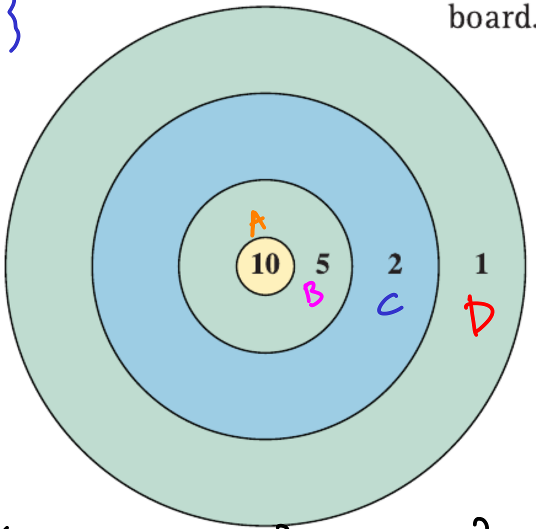
$B = [2,5]$

$$P(S \in B) = P(2 \le S \le 5) = \sum_{2 \le k \le 5} P(S = k) = p_S(2) + p_S(3) + \cdots + p_S(5)$$

$$= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} + \frac{4}{36} = \frac{5}{18}$$

Example 1.38. We have a dartboard of radius 9 inches. The board is divided into four parts by three concentric circles of radii 1, 3, and 6 inches. If our dart hits the smallest disk, we get 10 points, if it hits the next region then we get 5 points, and we get 2 and 1 points for the other two regions (see Figure 1.4). Let X denote the number of points we get when we throw a dart randomly (uniformly) at the board. How can we determine the distribution of X?

$X \in \{1, 2, 5, 10\}$



The radii of the four circles in the picture are 1, 3, 6 and 9 inches.

$$P_X(10) = P(X=10) = P(A) = \frac{\pi \cdot 1^2}{\pi \cdot 9^2} = 1/81$$

$$P_X(5) = P(X=5) = P(B) = \frac{\pi \cdot 3^2}{\pi \cdot 9^2} - \frac{\pi \cdot 1^2}{\pi \cdot 9^2} = \frac{8}{81}$$

$$P_X(2) = P(C) = \frac{\pi \cdot 6^2}{9^2} - \frac{\pi \cdot 3^2}{9^2} = \frac{1}{3}$$

$$P_X(1) = 1 - \sum_{k \neq 10} P_X(k) = 1 - \frac{27}{81} - \frac{1}{81} - \frac{8}{81} = 5/9$$

2.3 INDEPENDENCE (CONTD.)

INDEPENDENT RANDOM VARIABLES

Definition 2.27. Let $X_1, X_2, \ldots, X_n$ be random variables defined on the same probability space. Then $X_1, X_2, \ldots, X_n$ are independent if

$$P(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n) = \prod_{k=1}^{n} P(X_k \in B_k) \tag{2.14}$$

for all choices of subsets $B_1, B_2, \ldots, B_n$ of the real line. ♣

$X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n$

VERY COMPLEX !

$B_n = \mathbb{R}$

INDEPENDENCE FOR DISCRETE RVs

Fact 2.28. Discrete random variables $X_1, X_2, \ldots, X_n$ are independent if and only if

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{k=1}^{n} P(X_k = x_k) \qquad (2.15)$$

for all choices $x_1, x_2, \ldots, x_n$ of possible values of the random variables.

e.g. ROLL TWO DICE

$$\left.\begin{array}{ll} X_1 = & \text{OUTCOME OF 1st DIE} \\ X_2 = & \text{OUTCOME OF 2nd DIE} \\ S = & \text{SUM OF BOTH ROLLS} \end{array}\right\} \text{IND. ?}$$

X_1, X_2 $\quad$ S, X_1 $\quad$ $S, X_1, X_2 \rightarrow S = X_1 + X_2$

NOTE $X_k \in \{1, \dots, 6\}$

IF $i, j \in \{1, \dots, 6\}$

$$P(X_1 = i, X_2 = j) = P((i,j)) = \frac{1}{36}$$

$$\overset{\shortparallel}{P(X_1 = i) P(X_2 = j)} = \frac{6}{36} \cdot \frac{6}{36} = \frac{1}{36}$$

X_1 & X_2 ARE IND.

(i, j) $i = 1$

$i + j = 12$

$$\underset{\#}{P(X_1 = 1, S = 12)} = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{0}{36} = 0$$

$$P(X_1 = 1) P(S = 12) = \frac{1}{6} \cdot \frac{1}{36} \neq 0$$

X_1, S ARE NOT IND.

§ 2.4 INDEPENDENT TRIALS

A COMMON SITUATION: REPEATED INDEPENDENT TRIALS OF THE SAME EXPERIMENT

e.g. FLIPPING A COIN → H, T
ROLLING A DIE → $\{1, \ldots, 6\}$
SAMPLING W/ REPLACEMENT (ORDER MATTERS) → $\{1, \ldots, n\}$

GOAL: SYSTEMATICALLY STUDY A SIMPLIFIED MODEL.

$$\text{EXPERIMENT} \begin{cases} \nearrow \text{SUCCESS (DENOTED BY 1)} \\ \searrow \text{FAILURE (DENOTED BY 0)} \end{cases}$$

BASIC EXAMPLE: BERNOULLI RVs

NAME.

Definition 2.31. Let $0 \le p \le 1$. A random variable X has the $\boxed{\text{Bernoulli}}$ distribution with success probability p if X is $\{0, 1\}$-valued and satisfies $P(X = 1) = \boxed{p}$ and $P(X = 0) = 1 - p$. Abbreviate this by $X \sim \text{Ber}(p)$.

NOW SUPPOSE WE PERFORM 3 INDEPENDENT BERNOULLI TRIALS WITH THE SAME SUCCESS PROBABILITY

$$\Omega = \{(s_1, s_2, s_3) : s_j \in \{0,1\}\} = \{(0,0,0), (0,0,1), (1,1,0), \dots\} \to 2^8 \text{ OUTCOMES}$$

$$P(1,0,1) = P(X_1 = 1, X_2 = 0, X_3 = 1)$$

$$P(1,1,1)$$

$$P(1,0,1) = P(X_1=1, X_2=0, X_3=1)$$

ASSUMPTION

$$\boxed{=} \underbrace{P(X_1=1)}_{p} \cdot \underbrace{P(X_2=0)}_{1-p} \cdot \underbrace{P(X_3=3)}_{p}$$

$X_j : \Omega \to \mathbb{R}$

$X_j \to j^{th}$ TRIAL

$X_1(1,0,1) = 1$

$=$

$$P(1,0,1) = p \cdot (1-p) \cdot p = p^2 (1-p$$

$$P(1,1,1) = P(X_1=1, X_2=1, X_3=1)$$

$$= \prod_{j=1}^{3} \underbrace{P(X_j=1)}_{p} = p^3$$

$X \in \{0, 1\}$

BERNOULLI.

$p \rightarrow P(X = 1)$ SUCCESS.

$1 - p \rightarrow P(X = 0)$ FAILURE

IN GENERAL, n INDEP. BERNOULLI TRIALS

$$\Omega = \{ (s_1, \dots, s_n) : s_j = 0,1 \}$$

$\omega = (s_1, \dots, s_n)$

$$P(\omega) = p^{\#\, 1\text{s IN } \omega} \cdot (1-p)^{\#\, 0\text{s IN } \omega}$$

LET S = # OF SUCCESSES IN n BERNOULLI TRIALS

$$S = X_1 + X_2 + \dots + X_n, \qquad X_j \sim \text{Ber}(p) \text{ \& IND.}$$

$$P(S = k) = \sum_{\substack{\omega:\\ \omega \text{ HAS EXACTLY}\\ k\ 1\text{s}.}} P(\omega) = \#\{\omega : \text{EXACTLY } k\ 1\text{s}\}\, p^k (1-p)^{n-k}$$

$\omega = (s_1, s_2, \ldots, s_n) \qquad s_j = 0, 1$

$$P(S=k) = \#\{\omega : k \ 1\text{s}\} \cdot p^k (1-p)^{n-k}$$

$$(\overset{1}{\_}, \overset{*}{\_}, \overset{*}{\_}, \overset{1}{\_}, \ldots, \overset{*}{\_}) \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

$k \cdot 1\text{s} \qquad (n-k)\ 0\text{s}.$

$$P(S=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

p.m.f.

$s_1, \dots, s_n$

$$P(S=k) = \binom{n}{k} p^k (1-p)^{n-k}$$

VALID FOR $k \in \mathbb{Z}$, $0 \le k \le n$

$$P(S=k) = 0$$ IF $k < 0$ OR $k > n$ OR k IS NOT AN INTEGER.

CONV: $n \in \mathbb{N}$, $k \in \mathbb{Z}$

$$\binom{n}{k} = 0 \quad \text{IF} \quad n \notin \{0, 1, \dots, n\}$$

BINOMIAL RVs

DISCRETE

SUCCESS.

Definition 2.32. Let n be a positive integer and $0 \le p \le 1$. A random variable X has the **binomial distribution** with parameters n and p if the possible values of X are $\boxed{\{0, 1, \ldots, n\}}$ and the probabilities are

p.m.f.

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, \ldots, n.$$

Abbreviate this by $\boxed{X \sim \text{Bin}(n, p)}$.

NOTE: $\sum_{k=0}^{n} P(X = k) = \sum_{k=0}^{n} \binom{n}{k} \cdot p^k \underbrace{(1-p)}_{q}^{n-k}$

$= \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \underset{\text{BIN. THM.}}{=} (p+q)^n = 1$

$\sum_k \underbrace{P_X(k)}_{\text{p.m.f.}} = 1$

Example 2.33. What is the probability that five rolls of a fair die yield two or three sixes?

X = # OF SIXES IN 5 ROLLS

$X \sim \mathrm{Bin}(5, 1/6)$

$n = 5$

$p = 1/6$

$X = X_1 + X_2 + \cdots + X_5$

$X_j = \begin{cases} 1 & \text{IF ROLL } j = 6 \\ 0 & \text{o.w.} \end{cases}$

$P(X \in \{2,3\}) = P(X=2) + P(X=3) = \binom{5}{2}\left(\frac{1}{6}\right)^2 \cdot \left(1 - \frac{1}{6}\right)^3 + \binom{5}{3}\left(\frac{1}{6}\right)^3 \left(1 - \frac{1}{6}\right)^2$

$= 1500/7776 \approx 0.19$

GEOMETRIC RVs

CONSIDER NOW AN INFINITE SEQUENCE OF BERNOULLI TRIALS. $(X_1, X_2, X_3, \dots)$

$\hookrightarrow X_j \sim Ber(p)$, IND.

OBS: ANY EVENT WHICH ONLY CARES ABOUT THE FIRST k TRIALS IS ESSENTIALLY ON A FINITE SAMPLE SPACE $\longrightarrow$ CAN BE CALCULATED!

LET N = POSITION OF 1st SUCCESS WHEN DOING AN INFINITE SEQUENCE OF $Ber(p)$ TRIALS $(p \neq 0)$

$$P(N = k) = P(X_1 = 0, X_2 = 0, \dots, X_{k-1} = 0, X_k = 1)$$

$$P(N=k) = P(X_1=0, X_2=0, \dots, X_{k-1}=0, X_k=1)$$

$$= \underbrace{P(X_1=0)}_{1-p} \cdot \underbrace{P(X_2=0)}_{1-p} \cdots \underbrace{P(X_{k-1}=0)}_{1-p} \cdot \underbrace{P(X_k=1)}_{p}$$

$$= p\,(1-p)^{k-1}$$

$$P(N=k) = p\cdot(1-p)^{k-1}$$

DISCRETE

$k \in \{1, 2, 3, \dots\} = \mathbb{N}$

$N \in \mathbb{N} = \{1, \dots, \}$

Definition 2.34. Let $0 < p \leq 1$. A random variable X has the **geometric distribution** with success parameter p if the possible values of X are $\{1, 2, 3, \ldots\}$ and X satisfies $P(X = k) = (1-p)^{k-1}p$ for positive integers k. Abbreviate this by $X \sim \mathrm{Geom}(p)$.

p.m.f. GEOMETRIC PROGRESSION.

NOTE 1:

$$\sum_{k=1}^{\infty} \underbrace{P(X=k)}_{\text{p.m.f}} = \sum_{k=1}^{\infty} (1-p)^{k-1} p = \frac{p}{1-(1-p)}$$

FIRST TERM

COMMON RATIO

$$= 1$$

NOTE 2: $p = 0$?

T T T T T T . . . . .

$$\boxed{\begin{gathered} P(N=k) = 0 \\ P(N=\infty) = 1. \end{gathered}}$$

$\{1, \ldots\} \to \infty$

Example 2.35. What is the probability that it takes more than seven rolls of a fair die to roll a six?

N = POSITION OF FIRST SIX.

$$N \sim \text{Geom}\left(\frac{1}{6}\right)$$

p.m.f.

$$P(N > 7) = \sum_{k=8}^{\infty} P(N = k) = \sum_{k=8}^{\infty} \frac{1}{6} \cdot \left(1 - \frac{1}{6}\right)^{k-1}$$

$$= \frac{1/6 \cdot (5/6)^7}{1 - 5/6} = (5/6)^7$$

G.S. FORMULA

$$P(N > 7) = P(X_1 \neq 6, X_2 \neq 6, \ldots, X_7 \neq 6)$$

$$= \left(\frac{5}{6}\right)^7$$