MATH 201 (SUMMER 2023, SESH A2)

LECTURE 9 : 05/30/23

ANURAG SAHAY
OFF HRS: BY APPT (VIA ZOOM)

email: anuragsahay@rochester.edu

Zoom ID:
979-4693-0650

LECTURES:
9:00 AM – 11:15 AM (ET)
M, T, W, R

COURSE WEB PAGE: https://people.math.rochester.edu/grads/asahay/summer2023/math201/index.html

ALL PHOTOS TAKEN FROM TEXTBOOK

ANNOUNCEMENTS

1. MIDTERM 1 : THURS, 1st JUNE (IN-CLASS / SAMPLE / ~§4.4)
2. OFFICE HOURS : TW : 11:15 AM – 12:15 PM (+ BY APPT.)
3. UPCOMING DEADLINES :
   a. WW 05 – ~~TODAY~~, MAY ~~30th~~ 31st
   b. HW 04 – TODAY, MAY 30th → EXTENSION FOR O.H. ATTENDEES.
   c. WW 06 – ~~SAT~~, ~~JUNE 3rd~~ JUNE 6th
   d. HW 05 – SAT, JUNE 3rd
4. HW 1 IS GRADED. → SOLNS ARE UP
   → REGRADE REQUESTS.
5. PLEASE KEEP VIDEOS ON, IF POSSIBLE!

§ 4.2 LAW OF LARGE NUMBERS

FLIP A FAIR COIN N TIMES.

HOW MANY HEADS SHOULD SHOW UP AS $N \to \infty$?

$\sim N/2$ HEADS & $\sim N/2$ TAILS

NO HEADS $\leftarrow 2^{-N}$ $\qquad 2^{-N} \rightarrow N$ HEADS

[WEAK]

Theorem 4.8. (Law of large numbers for binomial random variables) *For any fixed* $\varepsilon > 0$ *we have*

$$\lim_{n\to\infty} P\left(\left| \frac{S_n}{n} - p \right| < \varepsilon \right) = 1. \tag{4.7}$$

w.h.p. $\frac{S_n}{n} \in (p-\epsilon, p+\epsilon)$

$S_n \sim \text{Bin}(n,p)$

$\frac{S_n}{n} \to$ FRACTION OF SUCCESSES

$$\left| \frac{\#\text{ OF HEADS}}{N} - \frac{1}{2} \right| < \epsilon$$

RELATION TO PREVIOUS SLIDE?

$S_n \sim Bin(n, 1/6)$

Example 4.9. Let S_n denote the number of sixes in n rolls of a fair die. Then S_n/n is the observed frequency of sixes in n rolls. Let $\varepsilon = 0.0001$. Then we have the limit

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \frac{1}{6}\right| < 0.0001\right) = 1.$$

The limit says that, as the number of rolls grows, deviations of S_n/n from 1/6 by more than 0.0001 become extremely unlikely. ▲

$$\left|\frac{S_n}{n} - \frac{1}{6}\right| < 0.0001 \qquad \text{w.h.p}$$

$$\Rightarrow \quad S_n \in \left(\left[\frac{1}{6} - 0.0001\right]n, \left[\frac{1}{6} + 0.0001\right]n\right)$$

Example 4.10. Show that the probability that fair coin flips yield 51% or more tails converges to zero as the number of flips tends to infinity.

$$P\left(\frac{\#\text{ OF HEADS}}{N} \geq \frac{51}{100}\right) \to 0 \quad \text{AS} \quad N \to \infty$$

$$S_n \sim \mathrm{Bin}(n, 1/2)$$

$$\frac{S_n}{n} \geq \frac{51}{100} \Rightarrow \frac{S_n}{n} - \frac{1}{2} \geq \frac{1}{100}$$

$$\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \frac{1}{100} \qquad \left[\frac{S_n}{n} \geq \frac{51}{100} \Rightarrow \left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \frac{1}{100}\right]$$

$$P\left(\frac{S_n}{n} \geq \frac{51}{100}\right) \leq P\left(\left|\frac{S_n}{n} - \frac{1}{2}\right| \geq \frac{1}{100}\right)$$

$$= 1 - \overbrace{P\left(\left|\frac{S_n}{n} - \underset{p}{\underbrace{\frac{1}{2}}}\right| < \underset{\epsilon}{\underbrace{\frac{1}{100}}}\right)}^{\text{L.L.N.} \to 1}$$

$$\lim_{n \to \infty} \text{R.H.S.} = 1 - 1 = 0$$

$$\lim_{n \to \infty} P\left(\frac{S_n}{n} \geq 51\%\right) = 0$$

C.L.T $\Rightarrow$ (WEAK) L.L.N.

$$\frac{S_n - \mathbb{E}(S_n)}{\sqrt{Var(S_n)}} = \hat{S}_n \approx Z \quad , \quad Z \sim N(0,1)$$

$$\Rightarrow \frac{S_n}{n} \approx \frac{\mathbb{E}(S_n)}{n} + \frac{\sqrt{Var(S_n)}}{n} Z$$

$S_n \sim Bin(n, p) \Rightarrow \mathbb{E}(S_n) = np$

$Var(S_n) = np(1-p)$

$$\frac{S_n}{n} \approx \frac{np}{n} + \frac{\sqrt{np(1-p)}}{n} Z \Rightarrow \frac{S_n}{n} - p \approx \underbrace{\sqrt{\frac{p(1-p)}{n}}}_{\to 0} \cdot Z$$

PF OF L.L.N. (SKIP PF)

LEMMA: FOR ANY $M > 0$, AND $n > \frac{\sigma^2 M^2}{\epsilon^2}$,

$$P\left(\left| \frac{S_n}{n} - p \right| < \epsilon \right) \geq P\left(-M < \hat{S}_n < M \right)$$

BIASED COIN, p → UNKNOWN

Q. HOW TO COMPUTE?

A: KEEP FLIPPING, COUNT # OF HEADS $(= S_n)$

$$\hat{p} = \frac{S_n}{n}$$

L.L.N. $\hat{p} \xrightarrow{?} \approx p$ w.h.p.

HOW BIG IS $|p - \hat{p}|$?

$P(|p - \hat{p}| < \epsilon)$ → CONFIDENCE

CLAIM: FOR $\epsilon > 0$,

$$P\left(|\tilde{\hat{p}} - \tilde{p}| < \epsilon \right) \geq 2\,\Phi\left(2\epsilon\sqrt{n}\right) - 1$$

PROVIDED n IS LARGE ENOUGH FOR THE NORMAL APPROXIMATION TO BE VALID.

OF COIN-FLIPS / TRIALS

Pf. $\hat{p} = \dfrac{S_n}{n}$ $\qquad \left| \dfrac{S_n}{n} - p \right| < \epsilon$

$$\left| S_n/n - p \right| < \epsilon$$

$$-\epsilon < \frac{S_n}{n} - p < \epsilon$$

$$-\epsilon < \frac{S_n - \underbrace{\boxed{np}}_{\mathbb{E}(S_n)}}{n} < \epsilon$$

$$-\epsilon\sqrt{\frac{n}{p(1-p)}} < \frac{S_n - \underbrace{\boxed{np}}_{\mathbb{E}}}{\sqrt{\underbrace{np(1-p)}_{Var}}} < \epsilon\sqrt{\frac{n}{p(1-p)}}$$

$$-\epsilon\sqrt{\frac{n}{p(1-p)}} < \hat{S}_n < \epsilon\sqrt{\frac{n}{p(1-p)}}$$

$\hat{S}_n \simeq N(0,1)$ [By C.L.T.)

$$P\left(\left|\frac{S_n}{n}-p\right|<\epsilon\right) = P\left(-\epsilon\sqrt{\frac{n}{p(1-p)}} < \hat{S}_n < \epsilon\sqrt{\frac{n}{p(1-p)}}\right)$$

$$\simeq P\left(-\epsilon\sqrt{\frac{n}{p(1-p)}} < Z < \epsilon\sqrt{\frac{n}{p(1-p)}}\right)$$

$$= \Phi\left(\epsilon\sqrt{\frac{n}{p(1-p)}}\right) - \Phi\left(-\epsilon\sqrt{\frac{n}{p(1-p)}}\right)$$

$\Phi(-x) = 1 - \Phi(x)$

$$= 2\,\Phi\left(\epsilon\sqrt{\frac{n}{p(1-p)}}\right) - 1$$

Ex.

$$\frac{1}{p(1-p)} \geqslant 4 \qquad (\text{For } p \in [0,1])$$

$$\Phi\left(\epsilon\sqrt{\frac{n}{p(1-p)}}\right) \geqslant \Phi\left(\epsilon\sqrt{4n}\right)$$

$$= \Phi(2\epsilon\sqrt{n})$$

$$P(|\hat{p}-p| < \epsilon) \geqslant 2\,\Phi(2\epsilon\sqrt{n}) - 1$$

Example 4.11. How many times should we flip a coin with unknown success probability p so that the estimate $\widehat{p} = S_n/n$ is within 0.05 of the true p, with probability at least 0.99?

$$P\left(|\hat{p} - p| < \underbrace{0.05}_{\epsilon}\right) \geq 0.99$$

WE KNOW,

$$P(|\hat{p} - p| < \epsilon) \geq 2\,\Phi(2\epsilon\sqrt{n}) - 1$$

$$\geq 2\,\Phi(2 \times 0.05\sqrt{n}) - 1$$

$$= 2\,\Phi\left(\frac{\sqrt{n}}{10}\right) - 1 \geq 0.99$$

$$\Phi\left(\frac{\sqrt{n}}{10}\right) \geqslant \frac{1+0.99}{2} = 0.995 = \Phi(2.58)$$

WHEN IS $\Phi(x) = 0.995$

2.58

$$\Rightarrow \frac{\sqrt{n}}{10} \geqslant 2.58$$

$$\Rightarrow n \geqslant 665.84$$

$(100\,r)\%$ CONFIDENCE INTERVAL : $(\hat{p}-\epsilon, \hat{p}+\epsilon)$

$r \in [0,1]$

s.t

$$P(|\hat{p}-p| < \epsilon) \geq r$$

Example 4.12. We repeat a trial 1000 times and observe 450 successes. Find the 95% confidence interval for the true success probability p.

$n = 1000$ $\qquad \hat{p} = \frac{450}{1000} = 0.45$

WHAT ϵ ENSURES THAT $\qquad$ ANS : $(\hat{p}-\epsilon, \hat{p}+\epsilon)$

$$P(|p-\hat{p}| < \epsilon) \geq \frac{95}{100}$$

$$P(|p - \hat{p}| > \epsilon) \geq 2\,\Phi(2\epsilon\sqrt{n}) - 1 \geq \frac{95}{100}$$

$$\Rightarrow \quad \underbrace{\Phi(2\epsilon\sqrt{n})} \geq \frac{1 + 0.95}{2} = 0.975$$

$$\downarrow$$

$$\Phi(2\epsilon\sqrt{1000}) \geq 0.975$$

$$\Phi(x) = 0.975 \Rightarrow x = 1.96$$

$$\Rightarrow 2\epsilon\sqrt{1000} \geq 1.96 \Rightarrow \epsilon \geq 1.96/2\sqrt{1000} \approx 0.031$$

$$\epsilon = 0.031$$

$$(\hat{p} - \epsilon, \hat{p} + \epsilon) = (0.45 - 0.031, 0.45 + 0.031)$$

$$= \underbrace{(0.419, 0.481)}_{\text{95\% CONFIDENCE INTERVAL}}$$

ANOTHER EXAMPLE: POLLING.

Example 4.14. Suppose that the fraction of a population who like broccoli is p. We wish to estimate p. In principle we could record the preferences of every individual, but this would be slow and expensive. Instead we take a random sample: we choose randomly n individuals, ask each of them whether they like broccoli or not, and estimate p with the ratio $\widehat{p}$ of those who said yes. We would like to quantify the accuracy of this estimate.

$$\hat{p} = \frac{\#\text{ WHO SAID YES}}{\#\text{ I ASKED}}.$$

Q: ARE SAMPLES INDEPENDENT? → NO!

= SAMPLING w/o REPLACEMT ≈ SAMPLING W/ REPLACEMENT (IF BAG IS VERY BIG)

$\epsilon = 0.05$, 90% CONF.

$n=100$, 90% CONF.

$$P(|\hat{p}-p|>\epsilon) \geq \frac{90}{100}$$

$$P(|\hat{p}-p|>\epsilon) \geq 2\Phi(2\epsilon\sqrt{n})-1$$

$$= 2\Phi(2\epsilon\sqrt{100})-1$$

$$= 2\Phi(20\epsilon)-1 \overset{?}{\geq} \frac{90}{100}$$

$$\Rightarrow \Phi(20\epsilon) \geq \frac{1+0.9}{2} = 0.95$$

$$\Phi(20\epsilon) \geq 0.95 \;\left(= \Phi(1.645)\right)$$

$$\Rightarrow 20\epsilon \geq 1.645$$

$$\Rightarrow \epsilon \geq \frac{1.645}{20} = 0.08225 \approx \underline{0.082}$$

90% CONFIDENCE INTERVAL $= (\hat{p} - \epsilon, \hat{p} + \epsilon) = (0.2 - 0.082, 0.2 + 0.082)$

$= (0.118, 0.282)$

$n = 100$, 20 → SAID YES $\quad \hat{p} = \frac{20}{100} = 0.2$

$\epsilon = 0.05$, 90% - CONFIDENCE

$$(\hat{p} - \epsilon , \hat{p} + \epsilon)$$

$$2\,\Phi(2\epsilon\sqrt{n}) - 1 \geq 0.9$$

$$\Phi(2\epsilon\sqrt{n}) \geq 0.95$$

$$\Rightarrow 2\epsilon\sqrt{n} \geq 1.645 \qquad (\Phi(1.645) = 0.95)$$

$$n \geq (1.645)^2 / (2 \times 0.05)^2 \approx 270.6$$

ANS : YOU NEED TO POLL

$$2711 = 1000 + \underbrace{1711}_{\text{EXTRA}}$$

Remark 4.15. (Confidence levels in political polls) During election seasons we are bombarded with news of the following kind: "The latest poll shows that 44% of voters favor candidate Honestman, with a margin of error of 3 percentage points." This report gives the confidence interval of the unknown fraction p that favor Honestman, namely $(0.44-0.03, 0.44+0.03) = (0.41, 0.47)$. The $\boxed{\text{level of confidence}}$ $\rightarrow$ OMITTED used to produce the estimate is usually omitted from news reports. This is no doubt partly due to a desire to avoid confusing technicalities. It is also a fairly common convention to set the confidence level at 95%, so it does not need to be stated explicitly. ▲

$\alpha = \frac{95}{100}$

POLLED $\leftarrow \hat{p} = 44\%$ $\epsilon = 0.03$

ACTUAL $\leftarrow p \in (\hat{p} - \epsilon, \hat{p} + \epsilon) = (0.44 - 0.03, 0.44 + 0.3)$

$= (0.41, 0.47)$

ASSUMPTION $\rightsquigarrow$ INDEPENDENT SAMPLES

(STATISTICAL POLLING)

BREAK TILL

10 :15 AM

§ 4.4 POISSON APPROXIMATION

FRENCH

Definition 4.17. Let $\lambda > 0$. A random variable X has the *Poisson distribution with parameter* λ if X is nonnegative integer valued and has the probability mass function

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad \text{for } k \in \{0, 1, 2, \dots\}. \tag{4.11}$$

Abbreviate this by $X \sim \text{Poisson}(\lambda)$. $/ X \sim \text{Pois}(\lambda)$

$\{0, 1, 2, 3, \dots\}$

(DISCRETE)

CHECK: ① $\sum$ p.m.f.

$$\sum_k P(X = k) = \sum_{k=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-\lambda} \underbrace{\boxed{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}}_{e^{\lambda}}$$

$$= e^{-\lambda} \cdot e^{\lambda} = 1$$

$$E(X) = \lambda$$

(2) MEAN

$$E(X) = \sum_{k} k \cdot P(X = k)$$

$$= \sum_{k=0}^{\infty} k \cdot \left(e^{-\lambda} \cdot \frac{\lambda^{k}}{k!} \right)$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \lambda^{k-1} \cdot \frac{\cancel{k}}{\cancel{k!}} \quad (k-1)!$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \underbrace{\boxed{e^{-\lambda} \sum_{j=0}^{\infty} \lambda^{j}/j!}}_{=1} = \lambda$$

③ $\mathbb{E}(X\cdot(X-1)) = \sum_{k} k(k-1)\, P(X=k)$

$$= \sum_{k=0}^{\infty} k(k-1)\, e^{-\lambda} \cdot \frac{\lambda^{k}}{k!}$$

$k=0,\ k=1 \longrightarrow$ NO CONT.

$$= \lambda^{2} e^{-\lambda} \sum_{k=2}^{\infty} \cancel{k\cdot(k-1)} \cdot \frac{\lambda^{k-2}}{\cancel{k!}\ (k-2)!}$$

$$= \lambda^{2} e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = \lambda^{2} e^{-\lambda} \sum_{j=0}^{\infty} \lambda^{j}/j! = \lambda^{2}$$

④ $\text{Var}(X) = \lambda$

$E(X) = \lambda$

$E(X\cdot(X-1)) = \lambda^2$

$$\text{Var}(X) = E(X^2) - E(X)^2$$

$$= E(X^2 - X + X) - E(X)^2$$

$$= E(X\cdot(X-1) + X) - E(X)^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2}$$

$$\sum_k [k\cdot(k-1) + k]\, P(X=k)$$

$$E(X\cdot(X-1)) + E(X)$$

$X \sim \text{Pois}$

p.m.f. $P(X = k) = e^{-\lambda} \dfrac{\lambda^k}{k!}$

MEAN $= \lambda$

VARIANCE $= \lambda$

LAW OF RARE EVENTS / POISSON APPROX.

SUPPOSE $X \sim \text{Bin}(n, p)$

WITH $n \longrightarrow \infty$ & $p = p(n)$ $\left[\longrightarrow 0 \text{ AS } n \longrightarrow \infty\right]$

C.L.T. DOES NOT APPLY.

$p(n) \longrightarrow p \neq 0$

e.g. $\mathbb{E}(X) = np = \text{FIXED } (:= \lambda) \Rightarrow p = \lambda/n$

$$\text{Bin}(n, \lambda/n) \xrightarrow[\text{DIST.}]{n \to \infty} \text{Pois}(\lambda)$$

LAW OF RARE EVENTS

Theorem 4.19. *Let $\lambda > 0$ and consider positive integers n for which $\underbrace{\lambda/n}_{p} < 1$. Let $S_n \sim Bin(n, \lambda/n)$. Then*

$$\lim_{n\to\infty} P(S_n = k) = e^{-\lambda}\frac{\lambda^k}{k!} \qquad \textit{for all} \quad k \in \{0, 1, 2, \dots\}. \tag{4.12}$$

$(S_n \sim Bin(n,p))$

Pf.

$$P(S_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \binom{n}{k} \frac{\lambda^k}{n^k} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \rightarrow (n)_k$$

$$P(S_n = k) = \frac{\boxed{(n)_k}}{\boxed{k!}} \cdot \frac{\boxed{\lambda^k}}{\boxed{n^k}} \left(1 - \frac{\lambda}{n}\right)^{n-k}$$

$$= \boxed{\frac{\lambda^k}{k!}} \cdot \underbrace{\boxed{\frac{(n)_k}{n^k}}}_{\to 1} \cdot \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\to e^{-\lambda}} \cdot \underbrace{\boxed{\left(1 - \boxed{\frac{\lambda}{n}}\right)^{-k}}}_{\to 1}$$

FIX k & λ & LET $n \longrightarrow \infty$.

n^k

$$\frac{n \cdot (n-1) \cdots (n-k+1)}{n^k} = \underbrace{\frac{\cancel{n}}{\cancel{n}}}_{1} \cdot \underbrace{\frac{(n-1)}{\cancel{n}}}_{1} \cdots \underbrace{\frac{(n-k+1)}{\cancel{n}}}_{1} \to 1$$

RECALL

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

$x = -\lambda$,

$$\lim_{n \to \infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$P(S_n = k) \longrightarrow e^{-\lambda} \cdot \frac{\lambda^k}{k!} \longrightarrow \text{p.m.f. of Poisson}(\lambda)$$

$E(Y) = E(X)$

Theorem 4.20. *Let* $X \sim \mathrm{Bin}(n,p)$ *and* $Y \sim \mathrm{Poisson}(np)$. *Then for any subset* $A \subseteq \{0,1,2,\ldots\}$, *we have*

$$|P(X \in A) - P(Y \in A)| \leq np^2. \tag{4.13}$$

$n \cdot p^2 \rightarrow$ SMALL

RECALL, $np(1-p) > 10 \rightarrow$ C.L.T.

$np^2 < 0.01 \rightarrow$ POISSON

Poisson approximation for counting rare events. Assume that the random variable X counts the occurrences of rare events that are $\boxed{\text{not strongly dependent}}$ on each other. Then the distribution of X can be approximated with a Poisson(λ) distribution for $\lambda = E[X]$. That is,

WEAKLY INDEPENDENT

$$P(X = k) \text{ is close to } e^{-\lambda}\frac{\lambda^k}{k!} \quad \text{for} \quad k \in \{0, 1, 2, \ldots\}. \tag{4.14}$$

Example 4.21. Suppose a factory experiences on average 3 accidents each month. What is the probability that during a particular month there are exactly 2 accidents?

① EVENT IS RARE.
② ACCIDENTS ARE INDEPENDENT.

$\Big\} \to$ POISSON (λ)

X = # OF ACCIDENTS IN A MONTH.

$\mathbb{E}(X) = 3 = \lambda$ ($\because$ POIS(λ) HAS MEAN λ)

$$P(X=2) = e^{-\lambda} \cdot \frac{\lambda^2}{2!} = e^{-3} \cdot \frac{3^2}{2!} = \frac{9e^{-3}}{2} \approx 0.224.$$

Example 4.22. The proofreader of an undergraduate probability textbook noticed that a randomly chosen page of the book has no typos with probability 0.9.

Estimate the probability that a randomly chosen page contains exactly two typos.

$E \sim \text{POISSON}(\lambda)$

$$P(E=0) = 0.9$$

$$\downarrow$$

$$e^{-\lambda} \cdot \frac{\lambda^{0}}{0!} = e^{-\lambda}$$

$$e^{-\lambda} = 0.9$$

$$\Rightarrow \lambda = -\ln(0.9)$$

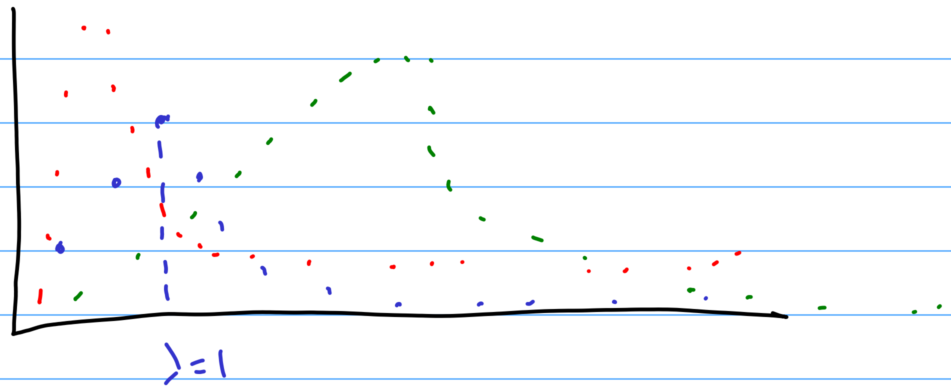
$$= \ln\left(\frac{10}{9}\right) \approx 0.105$$

$$P(x=2) = e^{-\lambda} \cdot \frac{\lambda^2}{2!} = e^{-(-\ln 0.9)} \cdot \frac{(0.105)^2}{2!}$$

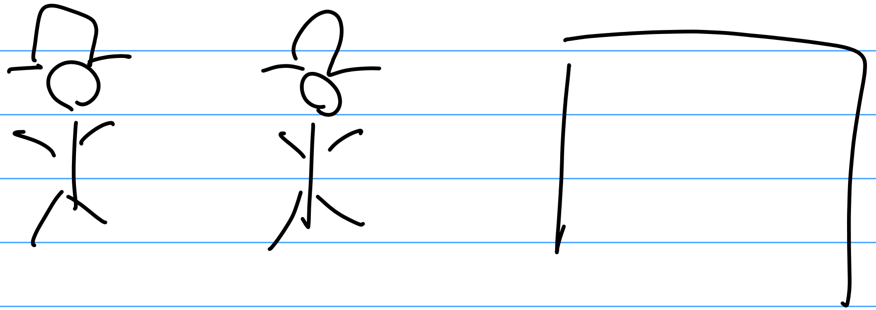
$$= \frac{0.9 \times (0.105)^2}{2}$$

$$\approx 0.005.$$

$$P(x=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$



Example 4.23. Consider again Example 1.27 of mixed-up hats with n guests. Let X denote the number of guests who receive the correct hat. How can we approximate the distribution of X? (AS $n \longrightarrow \infty$)



Ω — — — —

$(n-1)!$

$$P_r(\text{PERSON A GETS THEIR OWN HAT}) = \frac{\#\text{ FAVOR}}{\#\text{ TOTAL}} = \frac{(n-1)!}{n!} = \frac{1}{n}$$

PERSON A GETS OWN HAT

IS APPROX. INDEPENDENT

PERSON B GETS OWN HAT

A = PERSON A GETS OWN HAT

$$P(B \mid A) = \frac{1}{n-1}$$

$$P(B) = \frac{1}{n}$$

AS $n \to \infty$ $\quad P(B \mid A) \approx P(B)$

$X = \#$ WITH RIGHT HAT

$X \sim \mathrm{Pois}(\lambda)$

$$P\left(\begin{array}{c}\text{NO ONE GETS}\\ \text{THEIR OWN}\\ \text{HAT}\end{array}\right) \to \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} = e^{-1}$$

$$P(X=0) = e^{-1}$$

$$\| $$

$$e^{-\lambda} \cdot \frac{\lambda^0}{0!}$$

$$e^{-\lambda} = e^{-1} \Rightarrow \lambda = 1$$

$$P(X=k) = \frac{e^{-\lambda} \cdot \lambda^{k}}{k!} = \frac{e^{-1}}{k!}$$

END OF

PART A

(MIDTERM / FINAL : A SYLLABUS)