Math 201: Introduction to Probability

Midterm

June 1st, 2023

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- The exam will be 120 minutes long. You will get extra time in the end to upload the exam to Gradescope.
- There are 10 pages.
- A sheet with values of $\Phi(x)$ is provided.
- You may use any formulas from class without proof as long as you state it accurately **except where you are specifically asked for a proof**.
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You do not need to simplify complicated expressions such as $\binom{200}{15}$ or 500!.

QUESTION	VALUE	SCORE
1	0	
2	20	
3	20	
4	15	
5	20	
6	20	
7	20	
TOTAL	115	

Value Table of $\Phi(x)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ______________________________

2. (20 points)

Consider two urns, labelled H and T respectively. H contains two red balls. T contains five red balls, three blue balls and two green balls. Flip a coin and then uniformly draw two balls from the appropriate urn (H if heads, T if tails).

(a) What is the probability that you draw two green balls?

(b) What is the probability that you draw two red balls?

(c) What is the probability that you draw two blue balls?

(d) Given that you drew two balls of the same color, what is the probability that the coin flip turned up heads?

3. (20 points) Hades keeps rolling a fair, standard die until he rolls a 6. Let H be the number of times he rolls the die.

(a) H is an example of a random variable you have seen before. Identify the distribution and the parameters that specify the distribution.

(b) State the general formula for the mean and variance the of random variable you identified in part (a).

(c) Derive the formula for the variance you stated in part (b). (You may assume the formula for the expectation).

(d) What is the mean and variance of H?

4. (15 points) Two events A and B are called *mutually exclusive* if $P(AB) = 0$. Suppose A, B, and C are events such that and such that both of the following are true:

- $P(A) = 0.3, \qquad P(B) = 0.5, \qquad P(C) = 0.4.$
- A and C are mutually exclusive.
- A and B are independent.

Is it possible that B and C are mutually exclusive?

5. (20 points)

Define $f : \{0,1\}^2 \to \{0,1\}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $f(0,0) = f(1,1) = 0$ and $f(1,0) = f(0,1) = 1$. This is called the XOR operation.

Let X, Y be independent Bernoulli random variables with success probability $1/2$. Thus, X and Y are $\{0,1\}$-valued. Further, let Z be a random variable defined by

$$Z = f(X,Y).$$

(a) Show that X, Y, Z are pairwise independent.

(b) Are they mutually independent?

6. (20 points) Suppose $X \sim \text{Bin}(10, \frac{1}{10})$.

(a) Compute the value of $P(X \leqslant 1)$ exactly up to four digits. You may use the fact that $9^{10} \simeq 3.487 \times 10^9$ and $9^9 \simeq 3.874 \times 10^8$.

(b) Compute the Poisson approximation for $P(X \leqslant 1)$ up to four digits. You may use the fact that $2e^{-1} \simeq 0.7358$.

(c) Compute the Normal approximation without continuity correction for $P(X \leqslant 1)$ up to four digits.

(d) Compute the Normal approximation with continuity correction for $P(X \leqslant 1)$ up to four digits. You may use the fact that

$$\frac{0.5}{\sqrt{9/10}} \approx 0.53.$$

(e) Which of the above three approximations is the best?

7. (20 points) Let $Z \sim \mathcal{N}(0,1)$ be a standard normal variable, and let $X = e^Z$. Then, X is called a log-normal random variable.

(a) Express the c.d.f. of X in terms of $\Phi(t)$. Using this, compute $P(-2 \leqslant X \leqslant 1)$.

(b) Using your answer to part (a), explicitly compute the p.d.f. of X.

(c) Compute $E(X)$. [**Hint:** $-\frac{1}{2}x^2 + x = -\frac{1}{2}(x-1)^2 + \frac{1}{2}$.]