MATH 201 (SUMMER '23, SESH A2)

MIDTERM SOLUTIONS

2. (20 points)

Consider two urns, labelled H and T respectively. H contains two red balls. T contains five red balls, three blue balls and two green balls. Flip a coin and then uniformly draw two balls from the appropriate urn (H if heads, T if tails).

LET

R = TWO RED BALLS DRAWN

G = TWO GREEN BALLS DRAWN

B = TWO BLUE BALLS DRAWN

H = COIN-FLIP IS HEADS

T = COIN-FLIP IS TAILS

E = BOTH BALLS ARE THE SAME COLOUR.

(a) What is the probability that you draw two green balls?

OBSERVE THAT $T = H^c$, AND SO BY THE LAW OF TOTAL PROBABILITY,

$$P(G) = P(G|H)\cdot P(H) + P(G|T)\cdot P(T)$$

RECALL $P(\cdot|H)$ & $P(\cdot|T)$ ARE UNIFORM, HENCE IS GIVEN BY

$$\frac{\#\text{ FAVORABLE}}{\#\text{TOTAL}} \Rightarrow$$

① $P(G|H) = 0$ ($\because$ NO GREEN BALLS IN H)

② $P(G|T) = \frac{2\cdot 1}{10\cdot 9} = \frac{1}{45}$

[ASSUMING ORDER MATTERS ⇝ CAN USE ORDER DOESN'T MATTER ALSO]

SINCE $P(H) = P(T) = 1/2$,

$$P(G) = 0 \cdot 1/2 + 1/45 \cdot 1/2 = 1/90$$

(b) What is the probability that you draw two red balls?

THIS IS SIMILAR, EXCEPT THAT

$P(R \mid H) = 1$ ($\because$ BOTH BALLS IN H ARE RED)

$$P(R \mid H) = \frac{5 \cdot 4}{10 \cdot 9} = \frac{2}{9}$$

$$\therefore \quad P(R) = P(R \mid H)\,P(H) + P(R \mid T)\,P(T)$$

$$= 1 \cdot \tfrac{1}{2} + \tfrac{2}{9} \cdot \tfrac{1}{2} = \frac{11}{18}$$

(c) What is the probability that you draw two blue balls?

|||ly $$P(B) = P(B \mid H) \cdot P(H) + P(B \mid T)\,P(T)$$

$$= 0 \cdot \tfrac{1}{2} + \frac{3 \cdot 2}{10 \cdot 9} \cdot \tfrac{1}{2}$$

$$= \frac{1}{30}$$

(d) Given that you drew two balls of the same color, what is the probability that the coin flip turned up heads?

NOTE THAT

$$E = \underbrace{B \cup R \cup G}_{\text{PAIRWISE DISJOINT}}$$

$\Rightarrow$ BY ADDITIVITY, $P(E) = P(B) + P(R) + P(G)$

$$= \frac{1}{30} + \frac{11}{18} + \frac{1}{90} = \frac{59}{90}$$

O.T.O.H.

$$P(E|H) = \frac{P(E \cap H)}{P(H)} = \frac{P(H)}{P(H)} = 1$$

$(\because\ H \subseteq E \longrightarrow$ FLIPPING HEADS ALWAYS LEADS TO TWO BALLS OF THE SAME COLOUR SINCE ALL BALLS IN H ARE RED. $)$

$$\therefore P(H|E) = \frac{P(H \cap E)}{P(E)} = \frac{P(E|H) \cdot P(H)}{P(E)} = \frac{1 \cdot 1/2}{59/90} = \frac{45}{59}$$

(BAYES' FORMULA)

3. (20 points) Hades keeps rolling a fair, standard die until he rolls a 6. Let H be the number of times he rolls the die.

(a) H is an example of a random variable you have seen before. Identify the distribution and the parameters that specify the distribution.

GEOMETRIC $\longrightarrow$ $H \sim \text{Geom}(1/6)$

(b) State the general formula for the mean and variance the of random variable you identified in part (a).

IF $X \sim \text{Geom}(p)$ $\quad (q := 1-p)$

$$\mathbb{E}(X) = 1/p \quad , \quad \text{Var}(X) = \frac{q}{p^2} = \frac{1-p}{q^2}$$

(c) Derive the formula for the variance you stated in part (b). (You may assume the formula for the expectation).

$X \sim \text{Geom}(p)$

RECALL,

$$\text{Var}(X) = \mathbb{E}(X\cdot(X-1)) + \mathbb{E}(X) - \mathbb{E}(X)^2 \quad — (*)$$

$\mathbb{E}(X) = 1/p$, SO IT SUFFICES TO SHOW

$\mathbb{E}(X\cdot(X-1)) = 2q/p^2$, AS THEN R.H.S. OF (*) IS

$$\frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{1}{p^2}\left[2q + p - 1\right] = \frac{q}{p^2} \quad (\because p+q=1)$$

NOW,

$$E\left(X\cdot(X-1)\right) = \sum_{k} k\cdot(k-1)\; P(X=k)$$

$$= \sum_{k=1}^{\infty} k\cdot(k-1)\; q^{k-1}\, p \qquad \left(\text{p.m.f. OF } Geom(p)\right)$$

$$= \sum_{k=0}^{\infty} k\cdot(k-1)\; q^{k-1}\, p \qquad \left(\because\ k=0 \text{ TERM VANISHES}\right)$$

$$= pq \sum_{k=0}^{\infty} k\cdot(k-1)\; q^{k-2}$$

$$= pq \sum_{k=0}^{\infty} \frac{d^2}{dx^2}\left[x^k\right]_{x=q}$$

$$= pq \frac{d^2}{dx^2}\left[\sum_{k=0}^{\infty} x^k\right]_{x=q}$$

$(\because 0<q<1)$

$$= pq \frac{d^2}{dx^2}\left[\frac{1}{1-x}\right]_{x=q}$$

(GEOM. SERIES FORMULA)

$$= pq\left[\frac{2}{(1-x)^3}\right]_{x=q} = \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2} \quad \blacksquare$$

(d) What is the mean and variance of H?

$$p = \frac{1}{6} \Rightarrow q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\therefore \mathbb{E}(H) = \frac{1}{p} = 6$$

$$\mathrm{Var}(H) = \frac{q}{p^2} = \frac{5/6}{1/6^2} = 30$$

4. (15 points) Two events A and B are called *mutually exclusive* if $P(AB) = 0$. Suppose A, B, and C are events such that and such that both of the following are true:

- $P(A) = 0.3, \qquad P(B) = 0.5, \qquad P(C) = 0.4.$
- A and C are mutually exclusive. $\to$ $P(AC) = 0$
- A and B are independent. $\to$ $P(AB) = P(A)P(B) = (0.3)(0.5)$ $= 0.15$

Is it possible that B and C are mutually exclusive?

NO. SUPPOSE, FOR THE SAKE OF CONTRADICTION, THAT B & C ARE MUTUALLY EXCLUSIVE.

THEN, $P(BC) = 0$.

ALSO $P(ABC) \geq 0$ (BY DEFN.)

O.T.O.H.,

CLEARLY $ABC \subseteq AC$

$\Rightarrow P(ABC) \leq P(AC) = 0$

$\Rightarrow P(ABC) = 0$

BY INCLUSION-EXCLUSION

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$$

$$= 0.3 + 0.5 + 0.4 - 0.15 - 0 - 0 + 0 = 1.05$$

HENCE $P(A \cup B \cup C) = 1.05 > 1$

CONTRADICTION AS

PROB ≤ 1 ALWAYS

$\therefore$ B & C CANNOT BE MUTUALLY EXCLUSIVE.

Q5 & Q6 ARE IDENTICAL TO THE SAMPLE MIDTERM

$\Rightarrow$ SEE SOLNS. TO SAMPLE MIDTERM

7. (20 points) Let $Z \sim \mathcal{N}(0,1)$ be a standard normal variable, and let $X = e^Z$. Then, X is called a log-normal random variable.

(a) Express the c.d.f. of X in terms of $\Phi(t)$. Using this, compute $P(-2 \leqslant X \leqslant 1)$.

NOTE : $X > 0$ w/ PROB. $= 1$ AS $X = e^Z > 0$ ALWAYS.

$\therefore \; F_X(t) = 0$ IF $t \le 0$

OTOH, IF $t > 0$,

$$F_X(t) = P(X \le t) = P(e^Z \le t) = P(Z \le \ln t)$$

$$= \Phi(\ln t)$$ (BY DEFN OF Φ)

$$\therefore \quad F_X(t) = \begin{cases} 0, & t \leq 0 \\ \Phi(\ln t) & t > 0 \end{cases}$$

$$\therefore \quad P(-2 \leq X \leq 1) = F_X(1) - F_X(-2)$$

$$= \Phi(\ln 1) - 0$$

$$= \Phi(0) = 0.5$$

(b) Using your answer to part (a), explicitly compute the p.d.f. of X.

F_X IS CONTINUOUS

$\Rightarrow$ $f_X = F_X'$ ALMOST EVERYWHERE

IF $t < 0$, $f_X'(t) = 0$

CHAIN RULE

IF $t > 0$ $f_X'(t) = \frac{d}{dt}\Phi(\ln t) = \frac{d}{dt}(\ln t) \cdot \Phi'(\ln t)$

$= \frac{1}{t}\phi(\ln t) = \frac{1}{t}\left[\frac{e^{-(\ln t)^2/2}}{\sqrt{2\pi}}\right]$

$$\therefore \; f_X(t) = \begin{cases} 0, & \text{IF} \quad t < 0 \\ \dfrac{1}{t\sqrt{2\pi}} \exp\left(-\dfrac{(\ln t)^2}{2}\right) & \text{IF} \quad t > 0 \end{cases}$$

(c) Compute $E(X)$. **[Hint:** $-\frac{1}{2}x^2 + x = -\frac{1}{2}(x-1)^2 + \frac{1}{2}$.]

$$E(X) = \mathbb{E}\left(e^{Z}\right)$$

$$= \int_{-\infty}^{\infty} e^{t} \cdot \phi(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t} \cdot e^{-t^2/2}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2/2 + t}\, dt$$

BY HINT, $\dfrac{-t^2}{2} + t = \dfrac{-1}{2}(t-1)^2 + \dfrac{1}{2}$

$$\therefore E(X) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\left(-\frac{1}{2}(t-1)^2 + \frac{1}{2}\right)} dt$$

$$= e^{\frac{1}{2}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(t-1)^2} dt$$

SET $u = t - 1 \Rightarrow du = dt$

$$\therefore \ E(X) = e^{1/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du$$

BUT,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2/2} \, du = \int_{-\infty}^{\infty} \phi(u) \, du = 1$$

$(\because \phi$ IS A p.d.f.$)$

$$\therefore E(X) = e^{1/2} \cdot 1 = e^{1/2}$$