Math 201: Introduction to Probability

Sample Midterm

May 31st, 2023

NAME (please print legibly): ______________________________

Your University ID Number: ______________________________

- The exam will be 120 minutes long. You will get extra time in the end to upload the exam to Gradescope. 9:00 AM – 11:00 AM
- There are 10 pages.
- A sheet with values of $\boxed{\Phi(x)}$ is provided. → c.d.f. of a standard normal
- You may use any formulas from class without proof as long as you state it accurately **except where you are specifically asked for a proof**. $X \sim Bin(3, 1/3)$ $\mathbb{E}(X) = n \cdot p = 3 \cdot 1/3$
- No calculators, phones, electronic devices, books, notes are allowed during the exam. The only materials you are allowed to use are are pen/pencil and paper. In particular, you are NOT allowed to take the exam on a tablet.
- You are allowed to use a phone or tablet to take photographs of your answer sheet once the exam is over. If you finish early, you must take permission before taking photographs. Once you start taking photographs, you are not allowed to write.
- **Show all work and justify all answers as much as possible.** You may not receive full credit for a correct answer if insufficient work is shown or insufficient justification is given.
- You do not need to simplify complicated expressions such as $\binom{200}{15}$ or 500!. (UNLESS SPECIFIED)

e^{-5}

QUESTION	VALUE	SCORE
1	0	
2	25	
3	20	
4	15	
5	20	
6	20	
7	20	
TOTAL	120	

$X = 0.43$

Value Table of $\Phi(x)$

$\Phi(0.43)$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

1. (0 points) Copy the following honesty pledge on to your answer sheet. Remember to sign and date it.

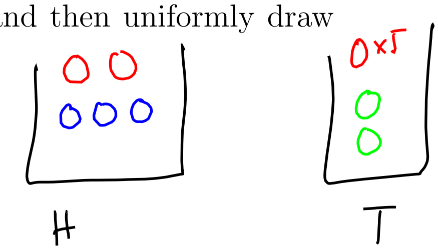
Pledge of Honesty

I affirm that I will not give or receive any unauthorized help on this exam, and that all work will be my own.

Signature: ____

2. (25 points)

Consider two urns, labelled H and T respectively. H contains two red balls and three ~~red~~ BLUE balls. T contains five red balls and two green balls. Flip a coin and then uniformly draw two balls from the appropriate urn (H if heads, T if tails).



(a) What is the probability that you draw two green balls?

LAW OF TOTAL PROB.

H = HEADS FLIPPED, T = TAILS FLIPPED

$H \cup T = \Omega$, $H \cap T = \phi$

$$P(2G) = P(2G|T)\cdot P(T) + \cancel{P(2G|H)\cdot P(H)} = \underbrace{P(2G|T)}_{\frac{2\cdot 1}{7\cdot 6}}\cdot(1/2)$$

(b) What is the probability that you draw two red balls?

$$P(2R) = \underbrace{P(2R|T)}_{\frac{5\cdot 4}{7\cdot 6}}\cdot \underbrace{P(T)}_{\frac{1}{2}} + \underbrace{P(2R|H)}_{\frac{2}{5\cdot 4}}\cdot \underbrace{P(H)}_{\frac{1}{2}}$$

(ORDER MATTERS)

$$\frac{\binom{2}{2}}{\binom{7}{2}}$$

[ORDER DOESN'T MATTER]

(c) What is the probability that you draw two blue balls?

$$P(2B) = P(2B|H)\cdot P(H) + P(2B|T)\cdot P(T)$$

$$= \frac{3\cdot 2}{5\cdot 4}\cdot \frac{1}{2} + 0 \cdot \frac{1}{2}$$

(d) Given that you drew two balls of different colors, what is the probability that the coin flip turned up heads?

BAYES' FORMULA

$$P(H \mid DIFF.) = \frac{P(H \cap DIFF.)}{P(DIFF.)}$$

$$= \frac{\underbrace{\boxed{P(H)}}_{1/2}\cdot \boxed{P(DIFF \mid H)}}{P(DIFF|H)\cdot P(H) + P(DIFF|T)\, P(T)}$$

$1 - P(SAME)$

$P(SAME) = P(2R) + P(2G) + P(2B)$

$= (a) + (b) + (c)$

$$P(DIFF \mid H) = \frac{3\times 2}{\binom{5}{2}}$$

$$= 3\times 2/5\cdot 2 = 3/5$$

3. (20 points) The Fates get up every morning and roll 3000 fair, standard dice and count the number of times they roll a number divisible by 3. Let F be this count.

(a) F is an example of a random variable you have seen before. Identify the distribution and the parameters that specify the distribution.

$$F \sim \text{Bin}(n, p)$$

$n = 3000$ $\quad$ $p = \frac{2}{6}$ SUCCESS PROBABILITY $= \frac{1}{3}$

(b) State the $\boxed{\text{general}}$ formula for the mean and variance the of random variable you identified in part (a).

$\text{Bin}(n, p)$

MEAN $= np$

VARIANCE $= np(1-p)$

(c) Derive the formula for the variance you stated in part (b). (You do not need to prove the formula for the expectation).

LOOK IN LECTURE NOTES 10.

(d) What is the mean and variance of F?

$$E(F) = n \cdot p = 3000 \cdot \frac{1}{3} = 1000$$

$$\text{Var}(F) = n \cdot p(1-p) = 3000 \cdot \frac{1}{3} \cdot \frac{2}{3}$$

$$= 2000$$

INCLUSION-EXCLUSION.

4. (15 points) Two events A and B are called *mutually exclusive* if $P(AB) = 0$. Suppose A, B, and C are events such that and such that both of the following are true:

$ABC \subseteq BC$

$\overset{\text{M of P}}{\Rightarrow}$ $P(ABC) \leq \cancel{P(BC)}$ 0

$\Rightarrow P(ABC) = 0$

- $P(A) = 0.1, \qquad P(B) = 0.8, \qquad P(C) = 0.4.$
- B and C are mutually exclusive. $P(BC) = 0$
- A and B are independent. $P(A \cap B) = P(A)\,P(B)$

$\Rightarrow P(AB) = 0.1 \times 0.8 = 0.08$

Is it possible that A and C are independent?

Q: $P(AC) = P(A)P(C) = 0.1 \times 0.4 = 0.04$

$P(B \cap C) = 0$

$P(B \cup C) = P(B) + P(C) - \cancel{P(B \cap C)}$

$= 0.8 + 0.4 = 1.2 > 1$

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - \underbrace{P(AB)}_{0.08} - \cancel{P(BC)} - \underbrace{P(AC)}_{0.04} + \boxed{\cancel{P(ABC)}}\ 0$$

$= 0.1 + 0.8 + 0.4 - 0.08 - 0.04$

$= 1.3 - 0.12 = 1.18 > 1$

#

5. (20 points)

$$f(y,x) = \begin{cases} 1 & y \neq x \\ 0 & \text{o.w.} \end{cases} = f(x,y)$$

Define $f : \{0,1\}^2 \to \{0,1\}$ by

IND. & FAIR. COIN-FLIPS

$$f(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $f(0,0) = f(1,1) = 0$ and $f(1,0) = f(0,1) = 1$. This is called the XOR operation.

Let $\boxed{X, Y}$ be independent Bernoulli random variables with success probability 1/2. Thus, X and Y are $\{0,1\}$-valued. Further, let Z be a random variable defined by

$$Z = f(X,Y). = \begin{cases} 1 & \text{IF } X \neq Y \\ 0 & \text{IF } X = Y \end{cases}$$

(a) Show that X, Y, Z are pairwise independent.

(b) Are they mutually independent?

① X & Y ARE IND.

② X & Z ARE IND

③ Y & Z ARE IND.

$X \leftrightarrow Y$

(BY SYMMETRY IN X & Y $f(x,y) = f(y,x)$)

DISCRETE R.Vs

$$P(X = k_1, Z = k_2) = P(X = k_1)\, P(Z = k_2)$$

$k_1, k_2 \in \{0,1\}$ $\qquad P(X = k_1) = 1/2 \quad k_1 \in \{0,1\}$

$$P(Z = 0) = P(f(X,Y) = 0) = P(X = Y) = P(X = Y = 1) + P(X = Y = 0) = \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = 1/2$$

$$P(Z = 1) = 1 - 1/2 = 1/2$$

$$P(X = 0, Z = 1) = P(X = 0, Y = 1) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

$\Updownarrow$

$X = 0, Y = 1$

6. (20 points) Suppose $X \sim \text{Bin}(10, \frac{1}{10})$. $X \in \{0, 1, \ldots, 10\}$

(a) Compute the value of $P(X \leqslant 1)$ exactly up to four digits. You may use the fact that $9^{10} \simeq 3.487 \times \underline{10}^9$ and $9^9 \simeq 3.874 \times \underline{10}^8$.

$$P(X \le 1) = P(X=0) + P(X=1)$$
$$= \sum_{k=0}^{1} \binom{n}{k} p^k \cdot (1-p)^{n-k} = \binom{10}{0} \cdot \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{10} + \binom{10}{1} \cdot \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^9$$
$$= 9^{10}/10^{10} + 9^9/10^{10} = \frac{9^{10} + 9^9 \times 10}{10^{10}} = \frac{(3.487 + 3.874)}{10}$$
$$\boxed{= 0.7361}$$

(b) Compute the Poisson approximation for $P(X \leqslant 1)$ up to four digits. You may use the fact that ~~$9^{10} \simeq 3.487 \times 10^9$~~ and ~~$9^9 \simeq 3.874 \times 10^8$~~.

$2e^{-1}$ = VALUE

$X \approx \text{Pois}(\lambda)$

$\lambda = n \cdot p = 10 \cdot 1/10 = 1$

$$P(X \le 1) = P(X=0) + P(X=1)$$
$$\approx \sum_{k=0}^{1} e^{-\lambda} \cdot \frac{\lambda^k}{k!} = e^{-1} \cdot \frac{1^0}{0!} + \frac{e^{-1} \cdot 1^1}{1!} = 2e^{-1} \approx$$

(c) Compute the normal approximation without continuity correction for $P(X \leqslant 1)$ up to four digits. $X \simeq N(\mu, \sigma^2)$

$$\hat{X} = \frac{X - E(X)}{\sqrt{\operatorname{Var} X}} = \frac{X-1}{\sqrt{9/10}}$$

$E(X) = 1$

$\operatorname{Var}(X) = 10 \cdot \frac{1}{10} \cdot \frac{9}{10} = \frac{9}{10}$

$$P(X \le 1) = P\left(\frac{X-1}{\sqrt{9/10}} \le 0\right) \approx \Phi(0) = 1/2 = 0.5$$

(d) Compute the normal approximation with continuity correction for $P(X \leqslant 1)$ up to four digits. You may use the fact that

$$\frac{0.5}{\sqrt{9/10}} \approx 0.53.$$

$$P(X \le 1) = P(0 \le X \le 1)$$
$$= P(0.5 \le X \le 1.5) = P\left(\frac{-0.5}{\sqrt{9/10}} \le \frac{X-1}{\sqrt{9/10}} \le \frac{0.5}{\sqrt{9/10}}\right)$$
$$\approx \Phi\left(\frac{0.5}{\sqrt{9/10}}\right) - \Phi\left(\frac{-0.5}{\sqrt{9/10}}\right)$$
$$\approx \Phi(0.53) - \Phi(-0.53)$$
$$= 2\boxed{\Phi(0.53)} - 1$$

(e) Which of the above three approximations is the best?

POISSON > NORMAL W/ CONT. CORR. >> NORMAL W/O CONT. CORRECTION.

7. (20 points) Let $Z \sim \mathcal{N}(0,1)$ be a standard normal variable, and let $X = Z^2$. Then, X is called a χ^2_1 random variable.

$P(X \leq 1) - \cancel{P(X < -2)}$

$= 2\Phi(1) - 1$

(a) Express the c.d.f. of X in terms of $\Phi(t)$. Using this, compute $\boxed{P(-2 \leqslant X \leqslant 1)}$.

0 IF $t < 0$

$$P(X \leq t) = P(Z^2 \leq t) =$$

$$Z^2 \leq t \iff -\sqrt{t} \leq Z \leq \sqrt{t}$$

$$= P(-\sqrt{t} \leq Z \leq \sqrt{t})$$

$$= \Phi(\sqrt{t}) - \Phi(-\sqrt{t})$$

$$= 2\Phi(\sqrt{t}) - 1$$

(b) Using your answer to part (a), explicitly compute the p.d.f. of X.

$$f(t) = \frac{d}{dt} F(t) = 0 \quad \text{IF } t < 0$$

$$= \frac{d}{dt}\left(2\Phi(\sqrt{t}) - 1\right) = 2 \cdot \left[\frac{1}{2} t^{-1/2}\right] \Phi'(\sqrt{t})$$

$$= t^{-1/2} \phi(\sqrt{t})$$

$$= \frac{t^{-1/2} e^{-(\sqrt{t})^2/2}}{\sqrt{2\pi}}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot t^{-1/2} e^{-t/2}$$

(c) Compute $E(X)$. [**Hint:** What is the variance of a standard normal?]

$$E(X) = \int_{-\infty}^{\infty} t \cdot f_X(t)\, dt$$

$$= \int_0^{\infty} t \cdot \left(\frac{1}{\sqrt{2\pi}} \cdot t^{-1/2} \cdot e^{-t/2}\right) dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} t^{1/2} \cdot e^{-t/2}\, dt$$

$$E(X) = E(Z^2) = \mathrm{Var}(Z^2) + E(Z)^2$$

$$= 1 + 0 = 1.$$

$Z \sim N(0,1)$

$E(Z) = 0$

$\mathrm{Var}(Z) = 1$

RECALL

$$\phi(t) = \left(e^{-t^2/2} \cdot 1/\sqrt{2\pi}\right)$$