Lecture 5

Definition 1. *(Free modules) Given a set T, we denote by $A^{(T)}$ the module*

$$\bigoplus_{t \in T} M_t$$

where each $M_t = A$. For each $t \in T$, we denote the canonical injection $A = M_t \to A^{(T)}$ by j_t, and we denote by e_t the element $j_t(1)$. Let $\phi: T \to A^{(T)}$ denote the mapping $t \mapsto e_t$.

Proposition 2. *(Universal property of $(A^{(T)}, \phi)$). For every A-module E and map $f: T \to E$, there exists one and only one linear map $g: A^{(T)} \to E$ such that*

$$f = g \circ \phi.$$

Definition 3. *A family $(a_t)_{t \in T}$ of elements of an A-module E is said to be a free family (respectively, a basis of E) if the linear map $A^{(T)} \to E$ determined by this family is injective (respectively, bijective). A module is called free if it has a basis.*

An element $x \in E$ is called free if $\{x\}$ is a free subset. It can happen that every non-zero element of a module is free without the module beong free (eg. $\mathbb{Q}$ as a $\mathbb{Z}$-module).

Proposition 4. *Every A-module is the quotient of a free module.*

Proposition 5. *Every exact sequence of A-modules*

$$0 \longrightarrow E' \longrightarrow E \longrightarrow F \longrightarrow 0$$

with F a free module splits.

Proposition 6. *If $A^{(I)} \cong A^{(J)}$, and either I or J is infinite, then $\operatorname{card} I = \operatorname{card} J$.*

The same is not necessarily true if I and J are both finite.

Definition 7. *(IBN property) A ring A is said to have the IBN (invariant basis number) property if whenever $A^m \cong A^n$, with $m, n \in \mathbb{N}$, we have $m = n$.*

Example 8. Many rings have the IBN property.

1. If there exists a ring homomorphism $\phi: R \to S$ and S has the IBN property then so does R.
2. From this it follows that all commutative rings have the IBN property (quotient by maximal ideals).
3. The following ring does not have the IBN property. Let V be an infinite dimensional k-vector space and let $A = \operatorname{End}_k(V)$. Then there exists an isomorphism $u: V \to V \oplus V$ (as A-modules). Applying the $\operatorname{Hom}_A(\cdot, V)$ functor to the exact sequence

$$0 \longrightarrow V \longrightarrow V \oplus V \longrightarrow 0$$

we obtain that $A \cong A \oplus A$, from which it follows that $A^m \cong A^n$ for any m, $n \in \mathbb{N}$. Note that, this is equivalent to saying that there exists matrices $P \in A^{m \times n}$, $Q \in A^{n \times m}$, such that $PQ = I_m$, $QP = I_n$.

Definition 9. *The annhilator of a subset S of an A-module E (denoted $\mathrm{Ann}(S)$) is the set of elements $a \in A$ such that $ax = 0$ for all $x \in S$. E is called faithful if* $\mathrm{Ann}(E) = 0$.

Proposition 10. *If $S \subset E$ is just a subset then $\mathrm{Ann}(S)$ is a left ideal of A. If $M \subset E$ is a sub-module then $\mathrm{Ann}(M)$ is a two-sided ieal of A.*

Definition 11. *Let E be an A-module and $\mathfrak{a} = \mathrm{Ann}(E)$. Then defining $(a + \mathfrak{a}) \cdot x = ax$, we get a $(A/\mathfrak{a})$-module structure on E. The $(A/\mathfrak{a})$-module thus defined is faithful, and is called the associated faithful module of E.*