Self maps of varieties over finite fields

A Thesis

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Synopsis

1 Background: Topological Entropy

A general reference for this section is the ICM article of Oguiso ([12]).

Let (X, d) be a compact metric space. Let $f : X \to X$ be a continuous self map of X. For $n \geqslant 1$, let (X^n, d_n) be a the n-times self-product of X equipped with the sup-metric induced by d. The continuous map $\Gamma_{f^{n-1}} : X \to X^n$ given by $x \to (x,\ f(x),\ \cdots, f^{n-1}(x))$, gives an embedding of $X \hookrightarrow X^n$. Let $d(f, n)$ be the metric induced on X by restriction of d_n under this embedding.

Intuitively, $d(f, n)$ measures how fast two points which were close to begin with, spread out or come closer as the case may be, under iteration by f. Let $N(\epsilon, n, f)$ be the least number of balls of radius ϵ with respect to $d(f, n)$, needed to cover X. Since X is compact, this is a finite number, which is non-decreasing as $\epsilon \to 0^+$ (for a fixed n).

Definition 0.0.1. With notations as above, the *topological entropy* of f denoted by $d_{\text{top}}(f) := \lim_{\epsilon \to 0^+} h(f, \epsilon)$, where $h(f, \epsilon) := \limsup_n \frac{\log N(\epsilon, n, f)}{n}$.

The limit above exists in $[0, \infty]$.

Following are the basic properties of topological entropy,

- $d_{\text{top}}(f)$ depends only on the underlying topology of X and not on the metric.
- If f is a periodic map then its entropy is 0.
- If f is an isometry then its entropy is 0.

An interesting class of compact metric spaces are compact Kahler manifolds with a choice of a Kahler metric.

Let M be a compact Kahler manifold and ω the associated $(1, 1)$ form. Let $f : M \to M$ be a holomorphic, surjective self map of a compact Kahler manifold. Then, as above to (M, f) we can associate $d_{\text{top}}(f) \in [0, \infty]$, the topological entropy of f.

Let $\lambda(f)$, $\lambda_{\text{even}}(f)$, $\lambda_p(f)$ denote the spectral radius for the (linear) action of f^* on $H^*(M, \mathbb{Q})$, $\oplus_i H^{2i}(M, \mathbb{Q})$ and $H^{p,p}(M, \mathbb{C})$. Then,

Theorem 0.0.2 (Gromov-Yomdin). *[1] [2]*
With notations as above, $d_{top}(f) = \log\lambda(f) = \log\lambda_{even} = \max_{0 \leqslant p \leqslant dim(X)} \log\lambda_p$.

Remark 0.0.3. Infact the proof of Gromov-Yomdin implies that the spectral radius for the action of f^* on $H^*(M, \mathbb{Q})$ is obtained on the smallest f^*-stable sub-algebra generated by any Kahler class ω.

Corollary 0.0.4. *The topological entropy of a surjective self map of a compact Kahler manifold is finite.*

Theorem 0.0.2 is computationally very useful and gives a 'simple' way to generate examples with positive entropy. Further, it linearizes the problem of computing topological entropy by relating it to the spectrum of f^* acting on cohomology. Hence, it is natural to look for constraints on the spectrum of this operator coming from various additional structures that can exist on cohomology.

2 Algebraic Entropy

The following proposition is a consequence of the existence of a polarized Hodge structure on $H^*(X(\mathbb{C}), \mathbb{Q})$ (with respect to any ample class $[\omega] \in H^2(X(\mathbb{C}), \mathbb{Q})$).

Proposition 0.0.5. *Let $X/\mathbb{C}$ be a smooth proper surface. Let $f : X \to X$ be an automorphism and $[\omega] \in H^2(X(\mathbb{C}), \mathbb{Q})$ an ample class. Then,*

1. *the spectral radius for the action of f^* on $H^*(X(\mathbb{C}), \mathbb{Q})$ coincides with the spectral radius for its action on the f^*-stable sub-algebra generated by $[\omega]$.*
2. *Moreover, f^* acts by finite order on $H^2_{tr}(X(\mathbb{C}), \mathbb{Q})$, the orthogonal complement (with respect to the cup-product pairing) of the image of Neron-Severi inside $H^2(X(\mathbb{C}), \mathbb{Q})$.*

The statement of the proposition above, makes sense over an arbitrary base field, with the Betti cohomology replaced by ℓ-adic cohomology. However, it is not suited for a proof by specialisation. Esnault and Srinivas observed that a suitable generalisation of Proposition 0.0.5 specialises well and proved the same by reduction to finite fields.

Theorem 0.0.6 (Esnault-Srinivas). *[3]*
Let $f : X \to X$ be an automorphism of a smooth proper surface over an arbitrary algebraically closed field k. Let ℓ be a prime invertible in k. Let $[\omega] \in H^2(X, \mathbb{Q}_\ell)$ be an ample class. Then for any embedding of $\mathbb{Q}_\ell$ inside $\mathbb{C}$,

1. *the spectral radius for the action of f^* on $H^*(X, \mathbb{Q}_\ell)$ coincides with the spectral radius for its action on the f^*-stable sub-algebra generated by $[\omega]$.*

2. *Moreover, let $V(f, [\omega])$ be the largest f^* stable sub-space of $H^2(X, \mathbb{Q}_\ell)$ in the orthogonal complement of $[\omega]$ (with respect to the cup-product pairing). Then f^* is of finite order on $V(f, [\omega])$.*

However, unlike Proposition 0.0.5 the proof of Theorem 0.0.6 is quite delicate and uses (among many other things) the explicit classification of smooth projective surfaces. In particular, it relies on lifting of certain $K3$ surfaces to characteristic 0 based on [4] and using Hodge theory to resolve this case.

Given the motivic nature of Theorem 0.0.6, it is natural to ask for analogues of the Gromov-Yomdin theory in positive characteristic (see [3] Section 6.2). This is the principal aim of this thesis.

Before we state the main results, we recall Varshavsky's formalism of contracting correspondences near sub-schemes (see [8]), a crucial ingredient for us.

3 Contracting correspondences near sub-schemes

Fix a base field k which is separably closed. Unless otherwise specified, all schemes are assumed to be of finite type and separated over k. Further, all morphisms of schemes are to be understood as over k.

Fix a prime ℓ invertible in k. Let Λ be a coefficient ring which is either finite and annihilated by a power of ℓ or is a finite extension of $\mathbb{Q}_\ell$ or is the ring of integers of such an extension.

Let $D^b_{ctf}(X, \Lambda)$ be the sub-triangulated category of the bounded derived category of sheaves of Λ-modules on X consisting of complexes of finite tor-dimension with constructible cohomology (see [23], 1.1.2). For a closed sub-scheme Z of X, let $\mathcal{I}_Z$ denote it's ideal sheaf. By Z_{red} we mean the reduced closed sub-scheme underlying Z. By $Z_d,\ d \geqslant 1$ we mean the closed sub-scheme of X defined by the ideal sheaf $\mathcal{I}_Z^d$. In particular $Z_1 = Z$ and Z_r is a closed sub-scheme of Z_s whenever $r \leqslant s$.

Definition 0.0.7 (Correspondence)**.** A *correspondence* from a scheme X_1 to X_2 is a morphism of schemes $c : C \to X_1 \times_k X_2$. We will denote this by $c = (C, c_1, c_2)$ where $c_1 := \mathrm{pr}_1 \circ c$ and $c_2 := \mathrm{pr}_2 \circ c$ with $\mathrm{pr}_i : X_1 \times_k X_2 \to X_i,\ i = 1, 2$ being the projections onto $X_i,\ i = 1, 2$.

Definition 0.0.8 (Cohomological correspondence)**.** Given a correspondence $c = (C, c_1, c_2)$ from X_1 to X_2 and objects $\mathcal{F}_i \in D^b_{ctf}(X_i, \Lambda)$ a *cohomological correspondence* (from $\mathcal{F}_1$ to $\mathcal{F}_2$) lifting c is a morphism $u \in \mathrm{Hom}_{D^b_{\mathrm{ctf}}(X_2,\Lambda)}(c_{2!}c_1^*\mathcal{F}_1, \mathcal{F}_2)$.

Definition 0.0.9. A closed sub-scheme Z is said to be *stabilized* by c if $c_2^{-1}(Z)$ is a closed sub-scheme of $c_1^{-1}(Z)$.

Definition 0.0.10. c is said to be *contracting* near a closed sub-scheme $Z \subset X$ if c stabilizes Z and $c_2^{-1}(Z_{n+1})$ is a closed sub-scheme of $c_1^{-1}(Z_n)$ for some $n \geqslant 1$.

For a correspondence c, let $\mathrm{Fix}(c)$ be the fibered product $C \times_{X \times_k X} X$ where, X is looked at as an $X \times_k X$-scheme via the diagonal embedding. Let c' be the induced morphism from $\mathrm{Fix}(c) \to X$.

Definition 0.0.11. A closed sub-scheme $Z \subset X$ is said to be *contracting in a neighbourhood of fixed points* if there exists an open sub-scheme W of C containing $\mathrm{Fix}(c)$ such that $c|_W$-contracting is contracting near Z.

Associated to a self-correspondence c of X and a cohomological self-correspondence u of $\mathcal{F} \in D^b_{\mathrm{ctf}}(X, \Lambda)$ lifting c and any $\beta \in \pi_0(\mathrm{Fix}(c))$ proper over k, one has a local term $\mathrm{LT}_\beta(u) \in \Lambda$ (see [8] 1.2.2).

The following result in [8] about the behaviour of local terms along contracted sub-schemes, is crucial for this work.

Theorem 0.0.12. *([8], Theorem 2.1.3)*

Let $c : C \to X \times_k X$ be a correspondence contracting near a closed sub-scheme $Z \subset X$ in a neighbourhood of fixed points, and let β be an open connected sub-set of $Fix(c)$ such that $c'(\beta) \cap Z \neq \varnothing$. Then

1. *β is contained set-theoretically in $c'^{-1}(Z)$, Hence β is an open connected subset of $Fix(c|_Z)$.*

2. *Suppose β is proper over k. Then, for every cohomological self-correspondence u of $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ lifting c, one has $LT_\beta(u) = LT_\beta(u|_Z)$.*

4 Main results

In this section unless otherwise mentioned, we work over $\mathbb{F}$, an algebraic closure of a finite field $\mathbb{F}_q$. Let $q = p^r$, where p is the characteristic of the finite field. Unless otherwise specified, all schemes are assumed to be separated and of finite type over $\mathbb{F}$.

A scheme (or a morphism of schemes) is said to be defined over $\mathbb{F}_q$ if it obtained by the base change to $\mathbb{F}$ of a scheme (or a morphism of schemes) over $\mathbb{F}_q$.

For any scheme $X_0/\mathbb{F}_q$, let $F : X_0 \to X_0$ be the r^{th}-iterate of the absolute Frobenius. We continue to denote by F the associated endomorphism of $X := X_0 \times_{\mathbb{F}_q} \mathbb{F}$.

Given a self-correspondence $c := (C, c_1, c_2)$ of a scheme X, both of which are defined over $\mathbb{F}_q$, denote by $c^{(n)}$ the self-correspondence $c^{(n)} := (C, c_1 \circ F^n, c_2)$ of X.

Let X be a scheme defined over $\mathbb{F}_q$. Let $j : X \hookrightarrow \overline{X}$ be a compactification and $\partial\overline{X}$ be the complement of X in $\overline{X}$, with the reduced induced structure. Assume that j (and hence $\partial\overline{X}$) is defined over $\mathbb{F}_q$. For a self-map $f : X \to X$, we denote by $\Gamma^t_f := (X, f, 1_X)$ the associated self-correspondence of X. For any proper self-map $f : X \to X$, let $\mathrm{Tr}(f^*, H^*_c(X, \mathbb{Q}_\ell)) := \sum_{i=0}^{2\dim(X)} (-1)^i \mathrm{Tr}(f^*, H^i_c(X, \mathbb{Q}_\ell)) \in \mathbb{Q}_\ell$.

To understand the action of a proper morphism on cohomology, it is necessary to be able to calculate the global traces of all its iterates. The following proposition ensures existence of good compactifications adapted to iteration,

Proposition 0.0.13. *Let f, g be two proper self maps of X defined over $\mathbb{F}_q$. Let $[\overline{f}]$ and $[\overline{g}]$ be compactifications of Γ^t_f and Γ^t_g. Suppose that $\partial\overline{X}$ is $[\overline{f}]^{(n)}$ and $[\overline{g}]^{(m)}$ contracting. Then, there exists a self-correspondence $[\widetilde{g \circ f}] := (\widetilde{C}_{g\circ f}, \widetilde{c}_{1,g\circ f}, \widetilde{c}_{2,g\circ f})$ of $\overline{X}$ and a morphism $[\widetilde{j}_{g\circ f}]$ from $\Gamma^t_{g\circ f}$ to $[\widetilde{g \circ f}]$ such that,*

1. *$[\widetilde{g \circ f}]$ is a self-correspondence of $\overline{X}$, also defined over $\mathbb{F}_q$.*

2. *$[\widetilde{g \circ f}]$ is a compactification of $\Gamma^t_{g\circ f}$ and $\widetilde{j}_{g\circ f,1} = \widetilde{j}_{g\circ f,2} = j$.*

3. *$\partial\overline{X}$ is $[\widetilde{g \circ f}]^{(m+n)}$-contracting.*

Moreover, $[\widetilde{g \circ f}]$ is independent of m or n.

Hence, we can deduce the following using Theorem 0.0.12.

Corollary 0.0.14. *Let $f : X \to X$ be a proper self map such that the pair (X, f) is defined over $\mathbb{F}_q$. There exists a $N(f) \geqslant 1$ such that for all integers $n \geqslant N(f)$ and $k \geqslant 1$, $Fix(f^k \circ F^{nk}) = Tr((f^k \circ F^{nk})^*, H^*_c(X, \mathbb{Q}_\ell))$, where $Fix(f^k \circ F^{nk})$ is the number of fixed points of $f^k \circ F^{nk}$ acting on X. Moreover, when X is proper we can take $N(f) = 1$.*

Now we study the consequences of these results to algebraic dynamics.

Let k be either $\mathbb{F}$ or the field of complex numbers $\mathbb{C}$. Fix an embedding $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$.

Suppose X is a proper scheme over k. Let $H^*(X)$ be the ℓ-adic cohomology of X (when $k = \mathbb{F}$) with its increasing weight filtration or the singular cohomology $H^*(X(\mathbb{C}), \mathbb{Q})$ (when $k = \mathbb{C}$) with its Mixed Hodge structure (see [5] section 2 and [25] Proposition 8.1.20). Let $W_k H^*(X)$ be the associated weight filtration.

Let $f : X \to X$ be a self-map of X.

Let λ_{odd} and λ_{even} be the spectral radius (with respect to τ, if $k = \mathbb{F}$) for the action of f^* on the oddly and evenly graded cohomology respectively (ℓ-adic or singular as the case may be). Let k_{odd} be maximal among integers with the property that, the spectral radius for the action of f^* on $\mathrm{Gr}^{k_{\text{odd}}}_W H^i(X)$ is λ_{odd}, where i is an odd integer. Similarly define k_{even}.

Theorem 0.0.15. *Using the above notations,*

1. *$\lambda_{even} \geqslant \lambda_{odd}$.*

2. *If equality holds in (1), then $k_{even} \geqslant k_{odd}$.*

As a result,

Corollary 0.0.16. *Let $f : X \to X$ be a self-map of a proper scheme over an arbitrary field k. Let ℓ be a prime invertible in k and $\overline{k}$ an algebraic closure of k. Fix an embedding $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Then the spectral radius (with respect to τ) for the action of $f^*_{\overline{k}}$ on the entire ℓ-adic cohomology $H^*(X_{\overline{k}}, \mathbb{Q}_\ell)$ is equal to the spectral radius for its action on the evenly graded cohomology.*

Moreover, we recover the following result of Fakhruddin [9] (see also [11]).

Theorem 0.0.17. *Let X_0 be a scheme over $k = \mathbb{F}_q$ and f_0 a proper, surjective self morphism of X_0. Then the set of periodic points of f_0 are Zariski dense in X_0.*

Organization of the thesis

This thesis consists of four chapters; we give a brief outline of the contents here.

In Chapter 1 we discuss the set-up underlying the Lefschetz-Verdier trace formula. In particular we define the local and global terms of the trace formula, and discuss some examples of interest. We also recall a trace formula due to Fujiwara and discuss some applications. Finally in Section 1.3 we define a *zeta function* associated to a self map of a variety over a finite field. We also establish some simple analytic properties of this zeta function.

Chapter 2 begins with a brief motivation for the problem considered in this thesis. In Section 2.1 we introduce the notion of *topological entropy* and discuss some foundational results by Gromov and Yomdin. In Section 2.2, as initiated by Esnault-Srinivas we attempt to understand the results of Section 2.1 from a motivic point of view. In particular we recall a result by Esnault-Srinivas on the structure of the linear map on the ℓ-adic cohomology, associated to an automorphism of a smooth proper surface. In Section 2.3 we establish some constraints on the eigenvalues for the action of a self-map of a proper scheme on its ℓ-adic cohomology. The crucial input here is Deligne's theory of weights.

In the scenario when the base field is either an algebraic closure of a finite field or the field of complex numbers, we put further restrictions on the action of such self maps on the graded pieces of the weight filtration. These results seems to be previously unknown even for smooth projective varieties over $\mathbb{C}$.

Finally in Section 2.4 we discuss some examples to elucidate the results of the previous section. The second example serves as a motivation for the question addressed in the next chapter.

In Chapter 3 we recall the notion of a *contracting correspondence* as introduced by Varshavsky and indicate some of its consequences. In particular we review an 'effective' trace formula for open correspondences obtained by Varshavsky. In Section 3.2 we construct compactifications adapted to both iteration and contraction. In Section 3.3 we use the compactifications constructed in the previous section to obtain a trace formula adapted to iterations, which is of independent interest.

In Chapter 4 we work exclusively with smooth, projective varieties. In Section 4.1 we review the necessary results from intersection theory. Then we establish an uniform bound for the intersection of subvarieties of complimentary dimension in a smooth projective variety. In the final section 4.2 we define the *Gromov algebra* associated to a self-map of a smooth projective variety and obtain a generalization of a result of Esnault-Srinivas, using an idea of O'Sullivan as developed by Truong [30].

Chapter 1

Lefschetz-Verdier trace formula

In this chapter we recall the set-up underlying the Lefschetz-Verdier trace formula. We essentially follow the treatment in [8]. We also recall a trace formula due to Fujiwara and discuss some applications of the same. In the final section we introduce a zeta function associated to a self map of a variety over a finite field.

In what follows we shall use basic properties of étale cohomology as developed in [19], [15]. A general reference for the framework of Verdier duality in the étale context is [13], Exposé XVIII. The passage to ℓ-adic coefficients is carried out in [14] and is summarized in [23], 1.1.

1.1 The set-up

Fix a base field k which is assumed to be separably closed. Unless otherwise specified, all schemes are assumed to be of finite type and separated over k. Further all morphisms of schemes are to be understood as over k. For any scheme X, 1_X denotes the identity morphism.

Fix a prime ℓ invertible in k. Let Λ be a coefficient ring which is either finite and annihilated by a power of ℓ, or is a finite extension of $\mathbb{Q}_\ell$, or is the ring of integers of such an extension. For most of our purposes, it suffices to consider $\Lambda = \mathbb{Q}_\ell$.

Let $D^b_{ctf}(X, \Lambda)$ be the bounded derived category of sheaves of Λ-modules on X, consisting of complexes of finite tor-dimension with constructible cohomology (see [15] Rapport 4.6, [23] 1.1.2-1.1.3). Let $D^b_{ctf}(\Lambda)$ denote the triangulated sub-category of the derived category of Λ-modules consisting of perfect complexes. For any scheme X, let Λ_X be the constant sheaf on X with coefficients in Λ.

For any scheme X, let $\pi_X : X \to \mathrm{Spec}\,(k)$ denote the structural morphism. Let $K_X := \pi_X^! \Lambda_{\mathrm{Spec}\,(k)}$, be the dualizing complex of X, and denote by $\mathbb{D}_X := \mathcal{RH}om(\ \ , K_X)$ the Verdier duality functor. For an embedding $f : Y \hookrightarrow X$ and any $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ we write $\mathcal{F}|_Y$ instead of $f^*\mathcal{F}$. We identify $D^b_{ctf}(\mathrm{Spec}\,(k), \Lambda)$ with $D^b_{ctf}(\Lambda)$ such that the functor $\pi_{X!}$ gets identified with $R\Gamma_c(X, \ \ )$.

For schemes X_1 and X_2, let pr_1 and pr_2 denote the projection morphisms from $X_1 \times_k X_2$ onto X_1 and X_2 respectively. Let $\mathcal{F}_i \in D^b_{ctf}(X_i, \Lambda),\ i = 1, 2$ be two complexes of sheaves. Denote by $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ the object $\mathrm{pr}_1^* \mathcal{F}_1 \otimes^L_\Lambda \mathrm{pr}_2^* \mathcal{F}_2$ in $D^b_{ctf}(X_1 \times_k X_2, \Lambda)$.

For a closed sub-scheme Z of X let $\mathcal{I}_Z$ denote it's ideal sheaf. By Z_{red} we mean the reduced closed sub-scheme underlying Z. By $Z_d,\ d \geqslant 1$ we mean the closed sub-scheme of X defined by the ideal sheaf $\mathcal{I}_Z^d$. In particular $Z_1 = Z$ and Z_r is a closed sub-scheme of Z_s, whenever $r \leqslant s$.

For a morphism of schemes $f : X \to Y$, (f^*, f_*) and $(f_!, f^!)$ are adjoint pairs. Further when f is proper we have an adjoint triple $(f^*, f_*, f^!)$.

For any scheme X, $\pi_0(X)$ denotes the set of its connected components.

Definition 1.1.1 (Correspondence)**.** A *correspondence* from a scheme X_1 to X_2 is a morphism of schemes $c : C \to X_1 \times_k X_2$. We will denote this by $[c] = (C, c_1, c_2)$ where $c_1 := \mathrm{pr}_1 \circ c$ and $c_2 := \mathrm{pr}_2 \circ c$.

Example 1.1.2. The natural isomorphism $c_{tr} : \mathrm{Spec}\,(k) \to \mathrm{Spec}\,(k) \times_k \mathrm{Spec}\,(k)$ is a self-correspondence of $\mathrm{Spec}\,(k)$, denoted by $[c_{tr}] = (\mathrm{Spec}\,(k), 1_{\mathrm{Spec}\,(k)}, 1_{\mathrm{Spec}\,(k)})$.

Example 1.1.3. Given a morphism of schemes $f : X_2 \to X_1$, we get a correspondence $[\Gamma^t_f] := (X_2, f, 1_{X_2})$ from X_1 to X_2. We identify X_2 with its image Γ^t_f inside $X_1 \times_k X_2$ via the correspondence $[\Gamma^t_f]$.

Definition 1.1.4 (Morphism of correspondences)**.** Let $[c] = (C, c_1, c_2)$ be a correspondence from X_1 to X_2 and let $[b] = (B, b_1, b_2)$ be a correspondence from Y_1 to Y_2. A *morphism of* $[c]$ *to* $[b]$ consists of a triple of morphisms $[f] := (f_1, f^\#, f_2)$ which make the following diagram commutative.

$$\begin{tikzcd} X_1 \arrow[d, "f_1"] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "f^\#"] & X_2 \arrow[d, "f_2"] \\ Y_1 & B \arrow[l, "b_1"] \arrow[r, "b_2"'] & Y_2 \end{tikzcd}$$

Example 1.1.5. Let $c : C \to X_1 \times_k X_2$ be a correspondence from X_1 to X_2. Then, $[\pi]_c := (\pi_{X_1}, \pi_C, \pi_{X_2})$ is a morphism from $[c]$ to $[c_{tr}]$ called the structural morphism of $[c]$.

We say a morphism of correspondences $[f] = (f_1, f^\#, f_2)$ is *proper* (resp. an *open immersion*, resp. a *closed immersion*) if each of the f_1, $f^\#$ and f_2 is proper (resp. an open immersion, resp. a closed immersion).

We say a correspondence $[c]$ is proper over k, if $[\pi]_c$ is proper.

Definition 1.1.6 (Compactification of correspondences)**.** A *compactification of a correspondence* $c : C \to X_1 \times_k X_2$, is an open immersion $[j] = (j_1, j^\#, j_2)$ of $[c]$ into a correspondence $\overline{c} : \overline{C} \to \overline{X}_1 \times_k \overline{X}_2$, such that $[\overline{c}]$ is proper and $j_1, j^\#, j_2$ are dominant.

Lemma 1.1.7. *Let $[\bar{c}]$ be a compactification of $[c]$ as above. If c_1 is proper, then the natural map $C \to \overline{C} \times_{\overline{X}_1} X_1$ is an isomorphism. Similarly, if c_2 is proper, the natural morphism $C \to \overline{C} \times_{\overline{X}_2} X_2$ is an isomorphism.*

Proof. Assume c_1 is proper, then the dense open immersion $C \hookrightarrow \overline{C} \times_{\overline{X}_1} X_1$ is also proper and hence, an isomorphism. A similar proof goes through when c_2 is proper. $\square$

Definition 1.1.8 (Restriction of a correspondence to an open sub-scheme)**.** Let $[c] = (C, c_1, c_2)$ be a correspondence from X to itself. Let $U \subseteq X$ be an open sub-scheme. Then the *restriction of c to U* is the correspondence,

$$[c]|_U := (c_1^{-1}(U) \cap c_2^{-1}(U), c_1|_{c_1^{-1}(U) \cap c_2^{-1}(U)}, c_2|_{c_1^{-1}(U) \cap c_2^{-1}(U)})$$

from U to itself. Let $c|_U$ denote the induced morphism from $c_1^{-1}(U) \cap c_2^{-1}(U) \to U \times_k U$.

Similarly if $W \subseteq C$ is an open sub-scheme of C, the restriction of C to W is the correspondence $[c]|_W := (W, c_1|_W, c_2|_W)$. As above $c|_W$ denotes the induced morphism from $W \to X \times_k X$.

Definition 1.1.9 (Cohomological correspondence)**.** Given a correspondence $[c] = (C, c_1, c_2)$ from X_1 to X_2 and objects $\mathcal{F}_i \in D^b_{ctf}(X_i, \Lambda)$ a *cohomological correspondence* (from $\mathcal{F}_1$ to $\mathcal{F}_2$) lifting $[c]$ is a morphism $u \in \mathrm{Hom}_{D^b_{ctf}(X_2,\Lambda)}(c_{2!}c_1^*\mathcal{F}_1, \mathcal{F}_2)$.

Example 1.1.10. Let $f : X_2 \to X_1$ be a morphism. Then, the natural isomorphism $f^*\Lambda_{X_1} \to \Lambda_{X_2}$ is a cohomological correspondence lifting $[\Gamma^t_f]$ (see Example 1.1.3).

Eventually we will be interested in cohomological correspondences lifting non-proper correspondences. To study these, it will be necessary to push-forward these cohomological correspondences to the compactifications of these non-proper correspondences. Conversely we will also need to pull-back cohomological correspondences lifting proper correspondences along an open sub-scheme (see Defintion 1.1.8).

1.1.1 Restriction of cohomological correspondence to an open sub-scheme

Let $c : C \to X \times_k X$ be a self-correspondence of X. Let $C^0 \subseteq C$ and $X_i^0 \subseteq X,\ i = 1, 2$ be open sub-schemes such that, c induces a correspondence $c^0 : C^0 \to X_1^0 \times_k X_2^0$. Then, one has the following commutative diagram,

$$\begin{tikzcd} C^0 \arrow[r, hook, "j_C"] \arrow[d, "c_2^0"] & C \arrow[d, "c_2"] \\ X_2^0 \arrow[r, hook, "j_{X_2^0}"] & X_2 \end{tikzcd}$$

For any sheaf $\mathcal{F}$ in $D^b_{ctf}(C,\Lambda)$, there exists a natural adjunction morphism $\mathcal{F} \to c_2^! c_{2!}\mathcal{F}$. Applying $j_C^!$ to the above morphism and using the adjunction between $c^0_{2!}$ and $c^{0!}_2$, one gets a base change morphism $BC(\mathcal{F}) : c^0_{2!}(\mathcal{F}|_{C^0}) \to (c_{2!}\mathcal{F})|_{X^0_2}$.

Let $u \in \mathrm{Hom}_{D^b_{ctf}(X_2,\Lambda)}(c_{2!}c_1^*\mathcal{F}_1, \mathcal{F}_2)$ be a cohomological correspondence lifting $[c]$. Then one can *restrict* u to give a cohomological correspondence from $\mathcal{F}_1|_{X^0_1}$ to $\mathcal{F}_2|_{X^0_2}$ lifting $[c_0]$ as follows,

$$u^0 : c^0_{2!}c^{0*}_1(\mathcal{F}_1|_{X^0_1}) \simeq c^0_{2!}(c_1^*\mathcal{F}_1|_{C^0}) \xrightarrow{BC(c_1^*\mathcal{F}_1)} (c_{2!}c_1^*\mathcal{F}_1)|_{X^0_2} \xrightarrow{u|_{X^0_2}} \mathcal{F}_2|_{X^0_2} .$$

In particular, for any open sub-scheme $U \subseteq X$ we have a cohomological correspondence $u|_U$ lifting $[c]|_U$ (see Definition 1.1.8).

1.1.2 Action of a correspondence on cohomology

Now suppose, one has a correspondence $c : C \to X_1 \times_k X_2$ and a cohomological correspondence u between $\mathcal{F}_1 \in D^b_{ctf}(X_1,\Lambda)$ and $\mathcal{F}_2 \in D^b_{ctf}(X_2,\Lambda)$ lifting $[c]$. Further assume that there exists an open sub-scheme $X^0_1 \subseteq X_1$ such that $\mathcal{F}_1$ is supported on X^0_1 and $c_1|_{c_1^{-1}(X^0_1)} : C^0 := c_1^{-1}(X^0_1) \to X^0_1$ is proper. Then, as shown in Section 1.1.1, one gets a correspondence $[c^0] := (C^0,\ c_1|_{C^0},\ c_2|_{C^0})$ between X^0_1 and X_2, and a cohomological correspondence u^0 between $\mathcal{F}_1|_{X^0_1}$ and $\mathcal{F}_2$ lifting $[c^0]$. Let c^0_1 and c^0_2 be the induced morphism from C^0 to X^0_1 and X_2 respectively.

Note that we have obvious isomorphisms $R\Gamma_c(X_1,\mathcal{F}_1) \simeq R\Gamma_c(X^0_1,\mathcal{F}_1|_{X^0_1}) = \pi_{X^0_1!}(\mathcal{F}_1|_{X_0})$ and $\pi_{C^0!}c^{0!}_2(\mathcal{F}_2) \simeq \pi_{X_2!}c^0_{2!}c^{0!}_2(\mathcal{F}_2)$. Since c^0_1 is proper one also has $\pi_{X^0_1!}c^0_{1*}c^{0*}_1(\mathcal{F}_1|_{X_0}) \simeq \pi_{C^0!}c^{0*}_1(\mathcal{F}_1|_{X^0_1})$. Further there is a morphism induced by adjunction $\pi_{X_2!}c^0_{2!}c^{0!}_2(\mathcal{F}_2) \to \pi_{X_2!}(\mathcal{F}_2)$. Thus applying $\pi_{C^0!}$ to u^0 gives a morphism $R\Gamma_c(u) : R\Gamma_c(X_1,\mathcal{F}_1) \to R\Gamma_c(X_2,\mathcal{F}_2)$. In particular if $X_1 = X_2$ and $\mathcal{F}_1 = \mathcal{F}_2$ then, one gets an endomorphism $R\Gamma_c(u)$ of the perfect complex $R\Gamma_c(X,\mathcal{F})$.

1.2 The trace formula

In this section we describe a recipe to obtain the local and global terms of the Lefschetz-Verdier trace formula. We continue using the notations and conventions of the previous section.

1.2.1 Scheme of fixed points

Let $c : C \to X \times_k X$ be a correspondence. The *scheme of fixed points* of $[c]$, is the closed sub-scheme $\mathrm{Fix}(c) := C \times_{X\times_k X} X$ of C, where X is looked at as a scheme over $X \times_k X$ via the diagonal embedding Δ. Let Δ' denote the embedding of $\mathrm{Fix}(c)$ inside C and let c' be the restriction of c to $\mathrm{Fix}(c)$. Note that $c_1 \circ c' = c_2 \circ c'$. Hence, we have a commutative diagram,

$$\begin{tikzcd}
\mathrm{Fix}(c) \arrow[r, "c'"] \arrow[d, hook, "\Delta'"] & X \arrow[d, hook, "\Delta"] \\
C \arrow[r, "c"] & X \times X
\end{tikzcd}$$

Example 1.2.1. Let $c : C \to X \times_k X$ be a correspondence from X to itself. Let u be a cohomological correspondence of $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ to itself lifting $[c]$. Further assume that c_2 is quasi-finite. The proper base change theorem ([15] Chapitre 1, Théorème 4.5.4) implies that for any closed point $x \in X$, the stalk at x of $c_{2!}c_1^*\mathcal{F}$ is isomorphic to $\oplus_{c_2(y)=x}\mathcal{F}_{c_1(y)}$. Hence u_x induces a morphism $\oplus_{c_2(y)=x}\mathcal{F}_{c_1(y)} \to \mathcal{F}_x$. In particular, for any closed point $y \in \mathrm{Fix}(c)$, there exists an induced endomorphism u_y of $\mathcal{F}_{c'(y)}$.

Suppose now, $[c] = [\Gamma^t_f]$ for a morphism $f : X \to X$. Then, the closed points of $\mathrm{Fix}(c)$ are precisely the 'fixed points' of f, that is closed points $y \in X$ such that $f(y) = y$. Further if u is the cohomological correspondence of Example 1.1.10, the induced endomorphism u_y of Λ, for any fixed point y of f, is the identity map of Λ.

Definition 1.2.2 (Naive local trace)**.** Using the assumptions and notations in Example 1.2.1, for any closed point $y \in \mathrm{Fix}(c)$, we define the *naive local term* at y to be the trace of the endomorphism u_y. We denote this by $\mathrm{NL}_y(u) \in \Lambda$.

As we shall see later, the Lefschetz-Verdier trace formula is a consequence of the commutativity of certain *trace maps* with proper push-forward. Now we describe these trace maps.

1.2.2 Trace maps

Let $c : C \to X \times_k X$ be a correspondence from X to itself. Let $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$. Let $\Delta : X \to X \times_k X$ be the diagonal embedding.

One has the natural evaluation map $\mathbb{D}_X\mathcal{F} \otimes \mathcal{F} \to K_X$. Since pullback commutes with (derived) tensor product, $\Delta^*(\mathbb{D}_X\mathcal{F} \boxtimes \mathcal{F}) \simeq \mathbb{D}_X\mathcal{F} \otimes \mathcal{F}$, by adjunction one gets a morphism $\mathbb{D}_X\mathcal{F} \boxtimes \mathcal{F} \to \Delta_* K_X$. Further one has the base change isomorphism (of functors) $c^!\Delta_* \simeq \Delta'_* c'^!$. Thus, applying $c^!$ to the morphism $\mathbb{D}_X\mathcal{F} \boxtimes \mathcal{F} \to \Delta_* K_X$ one gets a morphism $\phi_{\mathcal{F}} : c^!(\mathbb{D}_X\mathcal{F} \boxtimes \mathcal{F}) \to c^!(\Delta_* K_X) \simeq \Delta'_* c'^! K_X \simeq \Delta'_* K_{\mathrm{Fix}(c)}$, where the last isomorphism follows from the functoriality of the upper shriek functor.

In [17] (see (3.1.1) and (3.2.1) in loc. cit.), Illusie obtained a canonical isomorphism

$$\mathcal{RH}om(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \simeq c^!(\mathbb{D}_X\mathcal{F} \boxtimes \mathcal{F}).$$

Precomposing the above isomorphism with $\phi_{\mathcal{F}}$, we get a morphism

$$\underline{\mathcal{T}r} : \mathcal{RH}om(c_1^*\mathcal{F}, c_2^!\mathcal{F}) \to \Delta'_* K_{\mathrm{Fix}(c)}.$$

Applying $H^0(C, \quad)$ to the above morphism one obtains the *Trace map*

$$\mathcal{T}r_c : \mathrm{Hom}(c_{2!}c_1^*\mathcal{F}, \mathcal{F}) \to H^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}).$$

For an open subset β of $\mathrm{Fix}(c)$, let j_β denote the inclusion of β into $\mathrm{Fix}(c)$. The natural adjunction morphism $K_{\mathrm{Fix}(c)} \to j_{\beta *}j_\beta^* K_{\mathrm{Fix}(c)}$ induces a morphism

$$\mathrm{Res}_\beta : H^0(\mathrm{Fix}(c), K_{\mathrm{Fix}(c)}) \to H^0(\beta, K_\beta).$$

Denote by $\mathcal{T}r_\beta$ the composition $\mathrm{Res}_\beta \circ \mathcal{T}r_c$. If further β is proper over k, then the adjunction $\pi_{\beta !}\pi_\beta^! \Lambda \to \Lambda$ gives rise to a morphism $\mathrm{Int}_\beta : H^0(\beta, K_\beta) \to \Lambda$. Thus we get a morphism

$$LT_\beta := \mathrm{Int}_\beta \circ \mathcal{T}r_\beta : \mathrm{Hom}(c_{2!}c_1^*\mathcal{F}, \mathcal{F}) \to \Lambda.$$

Definition 1.2.3 (Local term)**.** For any proper connected component β of $\mathrm{Fix}(c)$ and any cohomological correspondence u lifting $[c]$, the *local term* at β is defined to be $LT_\beta(u)$.

Remark 1.2.4. Our definition of a local term is the one in [8], 1.2. It is compatible with the definition in [17], 4.2.5 (see [8], Appendix A).

Example 1.2.5. Let $[c] = [c_{tr}]$ as defined in Example 1.1.2. Recall that we have identified $D^b_{ctf}(\mathrm{Spec}\,(k), \Lambda)$ with the triangulated category of perfect complexes of Λ-modules. Moreover $\mathrm{Fix}(c_{tr}) = \mathrm{Spec}\,(k)$. Hence the trace map is a morphism from $Hom_{D^b_{ctf}(\Lambda)}(\mathcal{F}, \mathcal{F}) \to \Lambda$. The recipe above for the trace map implies that it coincides with the usual trace map for endomorphisms of perfect complexes.

Now we define the push-forward of a cohomological correspondence, in various situations which appear in our context.

1.2.3 Push-forward of cohomological correspondence

Let $[c] = (C, c_1, c_2)$ be a correspondence from X_1 to X_2 and $[b] := (B, b_1, b_2)$ a correspondence from Y_1 to Y_2. Let $[f] = (f_1, f^\#, f_2)$ be a morphism from $[c]$ to $[b]$(see Definition 1.1.4). Let $\mathcal{F}_i \in D^b_{ctf}(X_i, \Lambda),\ i = 1, 2$.

$$\begin{tikzcd}
X_1 \arrow[d, "f_1"] & C \arrow[l, "c_1"'] \arrow[r, "c_2"] \arrow[d, "f^\#"] & X_2 \arrow[d, "f_2"] \\
Y_1 & B \arrow[l, "b_1"] \arrow[r, "b_2"'] & Y_2
\end{tikzcd}$$

Suppose one of the following holds,

1. the left hand square is cartesian. Then there exists a base change isomorphism, $BC : b_1^* f_{1!} \to f^\#_! c_1^*$.

2. f_1 and $f^{\#}$ are proper. Since the left hand diagram is commutative, there exists a natural transformation $b_1^* f_{1*} \to f_*^{\#} c_1^*$ which by the properness assumption is the same as a natural transformation, $BC : b_1^* f_{1!} \to f_!^{\#} c_1^*$.

3. b_1 and c_1 are proper. By adjunction there is a natural transformation $f_{1!} \to f_{1!} c_{1*} c_1^* \simeq b_{1*} f_!^{\#} c_1^*$. Thus, in this case also, we get (by adjointness) a base change morphism, $BC : b_1^* f_{1!} \to f_!^{\#} c_1^*$.

Suppose at least one of the conditions above holds true, and let u be a cohomological correspondence between $\mathcal{F}_1$ and $\mathcal{F}_2$ lifting $[c]$. Then we can associate a cohomological correspondence $[f]_!(u)$ between $f_{1!}\mathcal{F}_1$ and $f_{2!}\mathcal{F}_2$ lifting $[b]$, as follows:

$$b_{2!} b_1^* (f_{1!}\mathcal{F}_1) \xrightarrow{BC} b_{2!} f_!^{\#} c_1^* \mathcal{F}_1 \simeq f_{2!} c_{2!} c_1^* \mathcal{F}_1 \xrightarrow{f_{2!}u} f_{2!}\mathcal{F}_2 .$$

Example 1.2.6. Let $[c] = (C, c_1, c_2)$ be a proper correspondence between X_1 and X_2. Then $[\pi]_c : [c] \to [c_{\mathrm{tr}}]$ satisfies the condition (2) above and hence $[\pi]_{c!}(u)$ makes sense for any cohomological correspondence u lifting $[c]$. Then it is immediate from definition that the push-forward as defined above, coincides with the action on cohomology defined earlier (see sub-section 1.1.2). Further, if $X_1 = X_2 = X$ and u is a cohomological self-correspondence of $\mathcal{F} \in D_{ctf}^b(X, \Lambda)$, then $[\pi]_{c!}(u)$ can be identified with the endomorphism $R\Gamma_c(u)$ of the perfect complex $R\Gamma_c(X, \mathcal{F})$.

Example 1.2.7. Suppose $f : X_2 \to X_1$ is a proper morphism of schemes. Let Γ_f^t be the correspondence from X_2 to X_1 associated to f (see Example 1.1.3). Then, the structural morphism $[\pi]_{\Gamma_f^t} : [\Gamma_f^t] \to [c_{\mathrm{tr}}]$ satisfies the condition (3) above. Hence one can push-forward the cohomological correspondence defined in Example 1.1.10. The recipe above implies that the induced morphism on cohomology is the obvious pullback on cohomology, $f^* : H_c^*(X_1, \Lambda) \to H_c^*(X_2, \Lambda)$ induced by f.

Example 1.2.8. Let $[c] := (C, c_1, c_2)$ be a correspondence between X_1 and X_2 with c_1 proper. Let $[\bar{c}] := (\overline{C}, \bar{c}_1, \bar{c}_2)$ be a compactification of $[c]$ via $[j] := (j_1, j^{\#}, j_2)$(see Definition 1.1.6). Since c_1 is proper, $[j]$ satisfies the condition (1) above (see Lemma 1.1.7). Thus for any $\mathcal{F}_i \in D_{ctf}^b(X_i, \Lambda)$, $i = 1, 2$ and any cohomological correspondence u between them lifting $[c]$, one gets a cohomological correspondence $[j]_!(u)$ between $j_{1!}\mathcal{F}_1$ and $j_{2!}\mathcal{F}_2$.

Suppose further that $X_1 = X_2 = X$ and u is a cohomological self-correspondence of $\mathcal{F} \in D_{ctf}^b(X, \Lambda)$. Then $[\pi]_c : [c] \to [c_{\mathrm{tr}}]$ satisfies condition (3) above and one gets an endomorphism $R\Gamma_c(u)$ of $R\Gamma_c(X, \mathcal{F})$, by pushing forward u along the structural map $[\pi]_c$. It is immediate from the definition of $[j]_!(u)$ that, this endomorphism is the same as the endomorphism $R\Gamma_c([j]_! u)$ of $R\Gamma_c(\overline{X},\ j_!\mathcal{F}) \simeq R\Gamma_c(X, \mathcal{F})$ obtained by pushing forward $[j]_!(u)$ (as defined above) along the structural map $[\pi]_{\bar{c}}$ (see Example 1.2.6).

1.2.4 Lefschetz-Verdier Trace formula

One can recover the Lefschetz-Verdier trace formula (see [17] Corollary 4.7) as a special case of commutativity of trace map with proper push-forward (see [17] Corollary 4.5, [8] 4.3.4).

Theorem 1.2.9. *Let $[c] = (C, c_1, c_2)$ be a correspondence from X to itself and $[b] = (B, b_1, b_2)$ be a correspondence from Y to itself. Let $[f] = (f, f^{\#}, f)$ be a proper morphism from $[c]$ to $[b]$. Then the morphism $f' : Fix(c) \to Fix(b)$ (see Definition 1.2.1) induced by $[f]$ is proper and for every cohomological correspondence u from $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ to itself, lifting $[c]$ one has,*

$$\mathcal{T}r_b([f]_!(u)) = f'_!(\mathcal{T}r_c(u)) \in H^0(Fix(b), K_{Fix(b)}).$$

Here $f'_! : H^0(Fix(c), K_{Fix(c)}) \to H^0(Fix(b), K_{Fix(b)})$ is the morphism induced by applying $H^0(Fix(b),\ )$ to the adjunction $f'_! K_{Fix(c)} \to K_{Fix(b)}$.

Remark 1.2.10. The theorem as stated is proved in [8], since we do not require c or b to be proper, unlike in [17].

An immediate corollary is the Lefschetz-Verdier trace formula.

Corollary 1.2.11. *Let $c : C \to X \times_k X$ be a correspondence with C and X proper over k. Then for every cohomological correspondence u from $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ to itself lifting c one has,*

$$Tr(R\Gamma_c(u)) = \textstyle\sum_{\beta \in \pi_0(Fix(c))} LT_\beta(u).$$

Here $Tr(R\Gamma_c(u))$ is the trace of the endomorphism $R\Gamma_c(u)$ of the perfect complex of Λ-modules $R\Gamma_c(X, \mathcal{F})$, induced by u (see Example 1.2.6).

Proof. The result follows from Theorem 1.2.9 applied to $[\pi]_c : [c] \to [c_{\mathrm{tr}}]$. Examples 1.2.6 and 1.2.5 imply that the term on the left (of Theorem 1.2.9) evaluates to $\mathrm{Tr}(R\Gamma_c(u))$. That the term on the right is $\sum_{\beta \in \pi_0(\mathrm{Fix}(c))} LT_\beta(u)$ follows from the definition of a local term (see Definition 1.2.3).

□

Given the Lefschetz-Verdier trace formula, the computation of global traces (for a proper correspondence) is reduced to the problem of computing local terms on the scheme of fixed points. Since the local terms are defined very non-explicitly, computing them in general can be quite difficult. However when one is working over an algebraic closure of a finite field, and under certain circumstances, these local terms can be computed explicitly. In fact under these circumstances, these local terms happen to be equal to the naive local terms (see Definition 1.2.2). Moreover the Lefschetz-Verdier trace formula can be used to compute the global traces (whenever they are defined) of cohomological correspondences lifting correspondences which are not necessarily proper. This was conjectured by Deligne and first proved (conditionally) by Pink ([7]) and unconditionally by Fujiwara ([6]).

1.3 Fujiwara's trace formula and applications

Let $k_0 = \mathbb{F}_q$ be a finite field with q elements of characteristic p. Let k be an algebraic closure of $\mathbb{F}_q$. Let ℓ be a prime co-prime to p.

Finite type and separated schemes over $\mathbb{F}_q$ will be denoted by a sub-script o (for example $X_0,\ Y_0$, etc.). Similarly morphisms of schemes over $\mathbb{F}_q$ will be denoted by $f_0,\ g_0$, etc. . The corresponding object over k will be denoted without a sub-script, for example $X,\ f$, etc. .

For any finite type and separated scheme X/k, let $H^i(X,\mathbb{Q}_\ell)$ and $H^i_c(X,\mathbb{Q}_\ell)$ respectively denote the i^{th} usual and compactly supported ℓ-adic étale cohomology of X.

For any self map $f : X \to X$ of a finite type and separated scheme X over k, we define

$$\mathrm{Tr}(f^*, H^*(X,\mathbb{Q}_\ell)) := \textstyle\sum_{i=0}^{2\dim(X)}(-1)^i\mathrm{Tr}(f^*; H^i(X,\mathbb{Q}_\ell)) \in \mathbb{Q}_\ell.$$

Similarly when f is proper, we define

$$\mathrm{Tr}(f^*, H^*_c(X,\mathbb{Q}_\ell)) := \textstyle\sum_{i=0}^{2\dim(X)}(-1)^i\mathrm{Tr}(f^*; H^i_c(X,\mathbb{Q}_\ell)) \in \mathbb{Q}_\ell.$$

Definition 1.3.1 (Absolute Frobenius)**.** For any scheme $X_0/\mathbb{F}_q$, the *absolute Frobenius* (with respect to $\mathbb{F}_q$) $F^a_{X_0,q} : X_0 \to X_0$ is the morphism which is the identity on the underlying topological space, and for any open affine sub-scheme $U_0 \subseteq X_0,\ F^a_{X_0,q}|_{U_0}$ corresponds to the ring endomorphism of $\Gamma(U_0,\mathcal{O}_{U_0})$ given by $\alpha \to \alpha^q,\ \forall \alpha \in \Gamma(U_0,\mathcal{O}_{U_0})$.

Remark 1.3.2. The absolute Frobenius is a $\mathbb{F}_q$-linear morphism.

Lemma 1.3.3. *Let $X_0/\mathbb{F}_q$ be a finite type and separated scheme. Then $X_0/\mathbb{F}_q$ is unramified (and hence étale) iff $F^a_{X_0,q}$ is an unramified morphism.*

Proof. The map induced on the differentials, $d : F^{a*}_{X_0,q}\Omega^1_{X_0/\mathbb{F}_q} \to \Omega^1_{X_0/\mathbb{F}_q}$ is the zero map. Hence $X_0/\mathbb{F}_q$ is an unramified morphism iff $\Omega^1_{X_0/\mathbb{F}_q}$ is the zero sheaf iff $X_0/\mathbb{F}_q$ is unramified (see [19] Chapter 1, Proposition 3.5).

□

Definition 1.3.4 (Geometric Frobenius)**.** For a scheme $X_0/\mathbb{F}_q$ let X denote the base change of X_0 to k. The *geometric Frobenius* (with respect to $\mathbb{F}_q$) is the morphism $F_{X,q} : X \to X$ induced from $F^a_{X_0,q} : X_0 \to X_0$ by base change.

Remark 1.3.5. The geometric Frobenius morphism is a k-linear morphism.

A scheme X/k is said to be *defined over* $\mathbb{F}_q$, if there exists a scheme $X_0/\mathbb{F}_q$ and an isomorphism of k-schemes between X and $X_0 \times_{\mathbb{F}_q} k$. Given any such scheme X/k defined over $\mathbb{F}_q$, the geometric Frobenius morphism $F_{X_0\times_{\mathbb{F}_q}k}$ induces an endomorphism of X/k. We call this the *geometric Frobenius* (with respect to $\mathbb{F}_q$) and denote it by $F_{X,q}$.

In a similar vein, a diagram of schemes over k is said to be defined over $\mathbb{F}_q$ if it is obtained as a base change of a diagram over $\mathbb{F}_q$.

Now suppose $f_0 : X_0 \to X_0$ is a self-map of a finite type and separated scheme over $\mathbb{F}_q$. Let (X, f) be the corresponding pair over k, obtained by base change. Let $\Gamma^t_{f_0 \circ F^a_{X_0,q}}$ be the transpose of the graph of $f_0 \circ F^a_{X_0,q}$ and Δ_{X_0} denote the diagonal embedding of X_0 in $X_0 \times_{\mathbb{F}_q} X_0$. Note that f_0 and $F^a_{X_0,q}$ commute.

Lemma 1.3.6. *With notations as above, the scheme $\Gamma^t_{f_0 \circ (F^a_{X_0,q})^m} \cap \Delta_{X_0}$ is étale over $\mathbb{F}_q$ for all $m \geqslant 1$.*

Proof. If m is greater than 1, replacing f_0 by $f_0 \circ (F^a_{X_0,q})^{m-1}$ we reduce to the case $m = 1$.

Let $Z_0 := \Gamma_{f_0 \circ F^a_{X_0,q}} \cap \Delta_{X_0}$ and $g_0 := f_0 \circ F^a_{X_0,q}$. Consider the diagram,

$$
\begin{tikzcd}
Z_0 \arrow[r, hook, "i"] \arrow[d, hook, "i"] & X_0 \arrow[d, hook, "\Delta"] & \\
X_0 \arrow[r, hook, "\Gamma^t_{g_0}"] \arrow[dr, "g_0"'] & X_0 \times_{\mathbb{F}_q} X_0 \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{pr}_2"] & X_0 \\
& X_0 &
\end{tikzcd}
$$

Here, the morphism $\Gamma^t_{g_0}$ is the transpose of the graph of g_0 and pr_i, $i = 1, 2$ are the projections. The commutativity of the above diagram implies that, $(f_0 \circ i) \circ F^a_{Z_0,q} = (f_0 \circ F_{X_0,q}) \circ i$ is a closed immersion. Hence $F^a_{Z_0,q}$ is a closed immersion and thus an unramified morphism. Lemma 1.3.3 now implies that Z_0 is unramfied (and hence étale) over $\mathbb{F}_q$.

□

Let $f : X \to X$ be a self-map of a finite type scheme over k.

Definition 1.3.7 (Fixed point)**.** A closed point $x \in X$ is said to be a fixed point of f if $f(x) = x$.

Proposition 1.3.8. *Let $f_0 : X_0 \to X_0$ be a self morphism of a smooth, proper scheme over $\mathbb{F}_q$. Then, $Tr((f^m \circ F^n_{X,q})^*, H^*(X, \mathbb{Q}_\ell))$ is the number of fixed points of $f^m \circ F^n_{X,q}$ acting on X.*

Proof. This is an immediate consequence of Lemma 1.3.6 and the trace formula in [15], Chapter 4, Corollaire 3.7.

□

The above proposition naturally leads to the following questions, for an arbitrary X_0, a proper f_0 and integers $m \geqslant 1$.

Question 1.3.9. Is the ℓ-adic number $\operatorname{Tr}\left((f \circ F^m_{X,q})^*, H^*_c(X, \mathbb{Q}_\ell)\right)$ an integer? If yes, then is it equal to the number of fixed points of $f \circ F^m_{X,q}$ acting on X?

A trace formula by Fujiwara sheds some light on these questions. We need one more definition before we can state Fujiwara's result.

Let C, X_1 and X_2 be finite type and separated schemes over k defined over $\mathbb{F}_q$. Let $[c] := (C, c_1, c_2)$ be a correspondence (see Definition 1.1.1) from X_1 to X_2 also defined over $\mathbb{F}_q$.

Definition 1.3.10. For every $n \geqslant 1$ let $[c]^{(n)} := (C, F^n_{X_1,q} \circ c_1, c_2)$, a correspondence from X_1 to X_2.

Remark 1.3.11. Note that $[c]^{(n)}$ is also defined over $\mathbb{F}_q$.

Suppose now that $X_1 = X_2 = X$ and c_1 is proper and c_2 is quasi-finite.

Theorem 1.3.12. *([6], Corollary 5.4.5)*

There exists an integer N_0 depending only on the correspondence $[c]$ such that for all integers $n \geqslant N_0$,

1. *$Fix(c^{(n)})$ is finite (as a scheme over k).*
2. *Moreover for any sheaf $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ and any cohomological self-correspondence u lifting $c^{(n)}$ one has,*

$$Tr(R\Gamma_c(u)) = \textstyle\sum_{x \in \pi_0(Fix(c^n))} NL_x(u).$$

Here $R\Gamma_c(u)$ is the endomorphism of $R\Gamma_c(X, \mathcal{F})$ as defined in Example 1.2.8 and $NL_x(u)$ is the naive local term at x (see Definition 1.2.2).

When X is proper we can take $N_0 = deg(c_2) := max_{x \in X}\left(dim_k H^0(c_2^{-1}(x), \mathcal{O}_{c_2^{-1}(x)})\right)$.

Now we are in a position to answer the questions 1.3.9.

Corollary 1.3.13. *Both the questions in 1.3.9 have a positive answer, if m is sufficiently large. Moreover when X_0 is proper any $m \geqslant 1$ would do.*

Proof. Let $[c] = [\Gamma^t_f]$. Then $[c]$ is defined over $\mathbb{F}_q$ and for any $n \geqslant 1$, the correspondence $[\Gamma^t_f]^{(n)} = [\Gamma^t_{f \circ F^n_{X,q}}]$. Let $\mathcal{F} = \mathbb{Q}_\ell$ and let $u^{(n)}$ be the cohomological self-correspondence of $\mathbb{Q}_\ell$ as in Example 1.1.10 lifting $[\Gamma^t_{f \circ F^n_{X,q}}] = [c]^{(n)}$. Then Fujiwara's trace formula (Theorem 1.3.12) implies that there exists an integer m such that, for all integers $n \geqslant m$,

$$\operatorname{Tr}\left((f \circ F^n)^*, H^*_c(X, \mathbb{Q}_\ell)\right) = \operatorname{Tr}(R\Gamma_c(u^{(n)})) = \textstyle\sum_{x \in \operatorname{Fix}(c^{(n)})} NL_x(u^{(n)}).$$

The first equality is a consequence of Example 1.2.7. That the right hand side computes the number of fixed point of $f \circ F^n_{X,q}$ is a consequence of Lemma 1.3.6 and Example 1.2.1. Moreoever we can choose $m = 1$ when X_0 is proper.

$\square$

Definition 1.3.14. Let $n_0(f)$ be the least integer m such that, both the questions in 1.3.9 have a positive answer.

Fujiwara's trace formula implies that for a non-proper X_0, questions 1.3.9 have a positive answer for m sufficiently large. An 'effective' upper bound for $n_0(f)$ is an immediate consequence of [8], Theorem 2.3.2 (see Corollary 3.1.14). However, there is no obvious relationship between these upper bounds for $n_0(f)$ and $n_0(f^r)$, $r > 1$. For the purpose of understanding the action of f^* on cohomology, it is useful to understand the growth of these upper bounds with respect to r. We shall address this question in Chapter 3.

We end this section with an interesting application of Fujiwara's trace formula (see [16], 3.5 b)).

Proposition 1.3.15. *Let $f : X \to X$ be a proper self map of a finite type and separated scheme over an arbitrary algebraically closed field k. Then for any prime ℓ invertible in k the ℓ-adic number $\sum_{i=0}^{2dim(X)} (-1)^i Tr(f^*; H^i_c(X, \mathbb{Q}_\ell))$ is in fact a rational number independent of ℓ.*

Proof. By a standard spreading out and specialisation argument, one is immediately reduced to the case where k is an algebraic closure of a finite field $\mathbb{F}_q$, and the pair (X, f) is defined over $\mathbb{F}_q$. Then Corollary 1.3.13 implies that there exists an integer N such that, the traces $\mathrm{Tr}\left(f \circ F^m_{X,q}, H^*_c(X, \mathbb{Q}_\ell)\right)$ are rational (in fact integral) for all integers $m \geqslant N$. The result is then an immediate consequence of Lemma 8.1 in [16] (see Remark 8.2, loc. cit.).

$\square$

Now we shall introduce a *zeta function* associated to a self map of a finite type and separated scheme over $\mathbb{F}_q$. The analytic properties of this zeta function will be central to the results in Chapter 2.

1.4 A zeta function associated to a self map

We continue using the notations and conventions from the previous sections.

Let X_0 be a finite type and separated scheme over $\mathbb{F}_q$ and $f_0 : X_0 \to X_0$ a proper self map (also defined over $\mathbb{F}_q$). Let ℓ be a prime invertible in $\mathbb{F}_q$.

For any field K, $K[[t]]$ and $K((t))$ respectively be the ring of formal power series and the field of formal Laurent series with coefficients in K.

Definition 1.4.1. The zeta function $Z(X_0, f_0, t)$ corresponding to (X_0, f_0) is defined to be

$$Z(X_0, f_0, t) = \exp(\textstyle\sum_{n\geqslant 1} \dfrac{\mathrm{Tr}((f \circ F_{X,q})^{n*}, H_c^*(X, \mathbb{Q}_\ell))t^n}{n}) \in \mathbb{Q}_\ell[[t]] \subset \mathbb{Q}_\ell((t)).$$

Remark 1.4.2. When f_0 is the identity, one recovers the Hasse-Weil zeta function of the scheme X_0.

Lemma 1.4.3. *$Z(X_0, f_0, t)$ belongs to $\mathbb{Q}(t) \cap \mathbb{Q}_\ell[[t]] \subset \mathbb{Q}_\ell((t))$ and is independent of ℓ.*

Proof. Proposition 1.3.15 implies that the above power series is actually in $\mathbb{Q}[[t]]$ and is independent of ℓ. The trace-determinant relation ([22] 1.5.3) implies that $Z(X_0, f_0, t)$ is a rational function in $\mathbb{Q}_\ell$. Hence, $Z(X_0, f_0, t) \in \mathbb{Q}_\ell(t) \cap \mathbb{Q}[[t]] \subset \mathbb{Q}_\ell[[t]] \subset \mathbb{Q}_\ell((t))$.

For any field K, a formal power series $\sum_{n\geqslant 0} a_n t^n \in K[[t]]$ is a rational function in t iff there exists N and M sufficiently large such that, for all $k > N$, the Hankel determinants $H_k := \det(A_k)$ vanish, where the matrix A_k is a $(M+1) \times (M+1)$ matrix and its $(i,j)^{\text{th}}$-entry is a_{i+j+k}, $0 \leqslant i, j, \leqslant M$ (see [18] Chapter 5, Section 5, Lemma 5, compare [19] Chapter VI, Remark 12.5). Hence for any field L containing K, $L(t) \cap K[[t]] = K(t) \cap K[[t]] \subset L((t))$ since, the vanishing of Hankel determinants corresponding to a formal power series can be checked after a field extension.

Thus $Z(X_0, f_0, t) \in \mathbb{Q}(t)$ and is independent of ℓ.

□

Corollary 1.4.4. *The formal power series $Z(X_0, f_0, t) \in \mathbb{Q}[[t]] \subset \mathbb{C}[[t]]$ has a non-trivial radius of convergence about the origin in the complex plane and has a meromorphic continuation onto the entire complex plane as a rational function.*

Proof. It follows from Lemma 1.4.3 that, the formal power series $Z(X_0, f_0, t)$ coincides with the power series expansion (about the origin) of a rational function with coefficients in $\mathbb{Q}$, as elements of $\mathbb{Q}[[t]]$. Hence the result.

□

Example 1.4.5. Suppose X_0 is geometrically connected and $f_0 : X_0 \to X_0$ is a constant map (necessarily mapping to a $\mathbb{F}_q$-valued point). The associated zeta function $Z(X_0, f_0, t)$ is $\frac{1}{1-t} \in \mathbb{Q}[[t]]$.

Chapter 2

Constraints on eigenvalues

In this chapter we establish some constraints on the eigenvalues for the linear action of a self map of a proper variety on its ℓ-adic cohomology. We begin with some motivation coming from topological entropy and then give a short summary of some recent work by Esnault-Srinivas ([3]). The results of this chapter partially answer a generalisation of some questions posed by Esnault-Srinivas (see loc. cit. 6.3).

2.1 Background: Topological entropy

A general reference for this section is the ICM article of Oguiso ([12]).

Let (X, d) be a compact metric space. Let $f : X \to X$ be a continuous self map of X. For $n \geqslant 1$, let (X^n, d_n) be a the n-fold self-product of X equipped with the sup-metric induced by d. The continuous map $\Gamma_{f^{n-1}} : X \to X^n$ given by $x \to (x,\ f(x),\ \cdots, f^{n-1}(x))$, gives an embedding of $X \hookrightarrow X^n$. Let $d(f, n)$ be the metric induced on X by restriction of d_n under this embedding.

Intuitively $d(f, n)$ measures how fast two points which were close to begin with, spread out or come closer as the case may be, under iteration by f. Let $N(\epsilon, n, f)$ be the least number of balls of radius ϵ with respect to $d(f, n)$, needed to cover X. Since X is compact, this is a finite number, which is non-decreasing as $\epsilon \to 0^+$ (for a fixed n).

Definition 2.1.1. With notations as above, the *topological entropy* of f denoted by $d_{\text{top}}(f) := \lim\limits_{\epsilon \to 0^+} h(f, \epsilon)$, where $h(f, \epsilon) := \limsup\limits_{n} \frac{\log N(\epsilon, n, f)}{n}$.

The limit above exists in $[0, \infty]$.

Following are the basic properties of topological entropy,

- $d_{\text{top}}(f)$ depends only on the underlying topology of X and not on the metric.
- If f is a periodic map then its entropy is 0.
- If f is an isometry then its entropy is 0.

An interesting class of compact metric spaces are compact Kahler manifolds with a choice of a Kahler metric.

Let M be a compact Kahler manifold and ω the associated $(1,1)$ form. Let $f : M \to M$ be a holomorphic, surjective self map of a compact Kahler manifold. Then as above to (M, f) we can associate $d_{\mathrm{top}}(f) \in [0, \infty]$, the topological entropy of f.

Let $\lambda(f)$, $\lambda_{\mathrm{even}}(f)$, $\lambda_p(f)$ denote the spectral radius for the (linear) action of f^* on $H^*(M, \mathbb{Q})$, $\oplus_i H^{2i}(M, \mathbb{Q})$ and $H^{p,p}(M, \mathbb{C})$. Then one has the following result obtained by combining results of Gromov [1] and Yomdin [2],

Theorem 2.1.2. *Let $f : M \to M$ be a holomorphic and surjective self map of a compact Kahler manifold M. With notations as above, $d_{top}(f) = \log \lambda(f) = \log \lambda_{even} = \max_{0 \leqslant p \leqslant dim(X)} \log \lambda_p$.*

Remark 2.1.3. In fact the proofs of Gromov and Yomdin imply that the spectral radius for the action of f^* on $H^*(M, \mathbb{Q})$ is obtained on the smallest f^*-stable sub-algebra generated by any Kahler class ω.

Corollary 2.1.4. *The topological entropy of a surjective self map of a compact Kahler manifold is finite.*

Theorem 2.1.2 is computationally very useful and gives a 'simple' way to generate examples with positive entropy. Further it linearizes the problem of computing topological entropy by relating it to the spectrum of f^* acting on cohomology. Hence it is natural to look for constraints on the spectrum of this operator, coming from the various additional structures that can exist on cohomology.

2.2 Algebraic entropy

The following proposition is a consequence of the existence of a polarized Hodge structure on $H^*(X(\mathbb{C}), \mathbb{Q})$ (with respect to any ample class $[\omega] \in H^2(X(\mathbb{C}), \mathbb{Q})$) (see [3] Proposition 5.1).

Proposition 2.2.1. *Let $X/\mathbb{C}$ be a smooth proper surface. Let $f : X \to X$ be an automorphism and $[\omega] \in H^2(X(\mathbb{C}), \mathbb{Q})$ an ample class. Then we have the following.*

(1) The spectral radius for the action of f^* on $H^*(X(\mathbb{C}), \mathbb{Q})$ coincides with the spectral radius for its action on the f^*-stable sub-algebra generated by $[\omega]$.

(2) Moreover f^* acts by finite order on $H^2_{tr}(X(\mathbb{C}), \mathbb{Q})$, the orthogonal complement (with respect to the cup-product pairing) of the image of Neron-Severi inside $H^2(X(\mathbb{C}), \mathbb{Q})$.

Here (1) can also be obtained as a consequence of the results of Gromov and Yomdin (see Remark 2.1.3).

The statement of the proposition above, makes sense over an arbitrary base field, with the Betti cohomology replaced by ℓ-adic cohomology. However it is not suited for a proof by specialisation. Esnault and Srinivas observed that a suitable generalisation of Proposition 2.2.1 specialises well, and proved the same by reducing to the case of a finite field (see [3]).

Theorem 2.2.2 (Esnault-Srinivas)**.** *Let $f : X \to X$ be an automorphism of a smooth proper surface over an arbitrary algebraically closed field k. Let ℓ be a prime invertible in k. Let $[\omega] \in H^2(X, \mathbb{Q}_\ell)$ be an ample class. Then for any embedding τ of $\mathbb{Q}_\ell$ inside $\mathbb{C}$,*

(1) the spectral radius for the action of f^* (with respect to τ) on $H^*(X, \mathbb{Q}_\ell)$ coincides with the spectral radius for its action on the f^*-stable sub-algebra generated by $[\omega]$.

(2) Moreover, let $V(f, [\omega])$ be the largest f^* stable sub-space of $H^2(X, \mathbb{Q}_\ell)$ in the orthogonal complement of $[\omega]$ (with respect to the cup-product pairing). Then f^* is of finite order on $V(f, [\omega])$.

Remark 2.2.3. The assumption, with respect to τ is superfluous, thanks to the result of Katz-Messing ([26] Theorem 2, (2)).

However, unlike Proposition 2.2.1 the proof of Theorem 2.2.2 is quite delicate and uses (among many other things) the explicit classification of smooth projective surfaces. In particular, it relies on lifting of certain $K3$ surfaces to characteristic 0, based on [4], and using Hodge theory to resolve this case.

2.3 Some new constraints on the eigenvalues

We use the notations and conventions of Section 1.3.

Lemma 2.3.1. *Let $G(t) \in t\mathbb{Q}[[t]]$ be a formal power series with non-negative coefficients and with no constant term. Then the formal power series $\exp(G(t)) \in \mathbb{Q}[[t]]$ and $G(t)$ have the same radius of convergence about the origin in the complex plane. In particular, in the disc of its convergence, the formal power series $\exp(G(t))$, considered as a holomorphic function, coincides with the exponential (in the analytic sense) of a holomorphic function.*

Proof. Suppose $G(t)$ converges in a disc of positive radius around the origin. Then it exists as a holomorphic function on the disc. Hence $F(t) := \exp(G(t))$ is a holomorphic function on this disc (here exp is the analytic exponential map).

Note that the power series expansion of the holomorphic function $F(t)$ about the origin (considered as an element of $\mathbb{Q}[[t]]$) coincides with the formal power series $\exp(G(t))$. Hence the formal power series $\exp(G(t))$ converges on the (open) disc of convergence of $G(t)$.

Hence the radius of convergence of the formal power series $\exp(G(t))$ is at least as large as that of the formal power series $G(t)$, subject to the latter having a non-trivial radius of convergence.

To complete the proof, it suffices to show that the radius of convergence of the formal power series $\exp(G(t))$ is bounded above by the radius of convergence of $G(t)$.

Using the standard expression for the radius of convergence of a formal power series (see [20] 10.5, (2)), it suffices to show that $\frac{d^n}{dt^n}(\exp(G(t)))|_{t=0} \geqslant G^{(n)}(0)$ where $G^{(n)}(t)$ is the n^{th} formal derivative of $G(t)$. We shall show by induction on n that,

$$\tfrac{d^n}{dt^n}(\exp(G(t)) = P(G^{(1)}(t), G^{(2)}(t), \ \cdots, G^{(n)}(t)) \exp(G(t)) \text{ in } \mathbb{Q}[[t]],$$

where $P(x_1, x_2, \cdots, x_n)$ is a polynomial with *positive integral* coefficients and is of the form $P(x_1, x_2, \cdots, x_n) = x_n + \widetilde{P}(x_1, x_2, \cdots, x_{n-1})$ for some polynomial $\widetilde{P}$ in one less variable.

For $n = 1$ the statement is obviously true. Assume now that the statement is true with $n = k$, the chain rule of differentiation then implies the statement for $n = k + 1$. In particular, one observes that

$$\tfrac{d^n}{dt^n}(\exp(G(t)))|_{t=0} = G^{(n)}(0) + \widetilde{P}(G^{(1)}(0), G^{(2)}(0), \cdots, G^{(n-1)}(0)).$$

The non-negativity of the coefficients of $G(t)$ implies the non-negativity of $G^{(n)}(0)$ for each $n \geqslant 1$. Since the coefficients of the polynomial $\widetilde{P}$ are positive we are done.

$\square$

Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$ and let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q} \subset \mathbb{Q}_\ell$ inside $\overline{\mathbb{Q}}_\ell$. An element in $\overline{\mathbb{Q}}_\ell$ is said to be an algebraic number, if it belongs to $\overline{\mathbb{Q}}$.

The following proposition on existence of functorial weight filtrations is standard. We prove it here due to lack of an appropriate reference.

Proposition 2.3.2. *Let X_0 be a finite type and separated scheme over a finite field $\mathbb{F}_q$. For any integer $i \in [0, 2dim(X)]$, there exists an unique increasing (finite) weight filtration $W_k H^i_c(X, \mathbb{Q}_\ell)$, $k \geqslant 0$ such that,*

1. *(Functoriality) the weight filtration is functorial for proper morphisms of finite type and separated schemes over $\mathbb{F}_q$. That is, for any proper morphism $f_0 : X_0 \to Y_0$ of finite type and separated schemes over $\mathbb{F}_q$, the induced morphism $f^* : H^i_c(Y, \mathbb{Q}_\ell) \to H^i_c(X, \mathbb{Q}_\ell)$ respects the weight filtration. In particular there is an induced action of $F^*_{X,q}$ on $Gr^k_W H^i_c(X, \mathbb{Q}_\ell)$.*
2. *(Purity) For any integer $i \in [0, 2dim(X)]$ and all integers $k \geqslant 0$, the non-zero $Gr^k_W H^i(X, \mathbb{Q}_\ell)$ are pure of weight k. That is, for any integer $i \in [0, 2dim(X)]$ and all integers k, the eigenvalues (in $\overline{\mathbb{Q}}_\ell$) of $F^*_{X,q}$ acting on $Gr^k_W H^i_c(X, \mathbb{Q}_\ell)$ (assumed to be non-zero) are algebraic numbers, all of whose complex conjugates have absolute value $q^{\frac{k}{2}}$.*

(3) *(Strictness) For any proper morphism $f_0 : X_0 \to Y_0$ of finite type and separated schemes over $\mathbb{F}_q$, the induced morphism $f^* : H^i_c(Y, \mathbb{Q}_\ell) \to H^i_c(X, \mathbb{Q}_\ell)$ is strict (in the sense of [24] 1.1.5) for the weight filtration.*

Proof. Let $\mathcal{A}$ be the abelian category of $\mathbb{Q}_\ell[t]$-modules. For any finite type and separated scheme $X_0/\mathbb{F}_q$, $H^i_c(X, \mathbb{Q}_\ell)$ can be considered as a $\mathbb{Q}_\ell[t]$-module with t acting on $H^i_c(X, \mathbb{Q}_\ell)$ as $F^*_{X,q}$.

Let $\mathcal{C}_{\mathbb{F}_q}$ be the category whose objects are finite type and separated schemes over $\mathbb{F}_q$ and morphisms being proper morphisms of schemes over $\mathbb{F}_q$. Then for any i, $X_0 \mapsto H^i_c(X, \mathbb{Q}_\ell)$ is a (contravariant) functor from $\mathcal{C}_{\mathbb{F}_q}$ to $\mathcal{A}$.

Théorème 1 (3.3.1) in [23] implies that the eigenvalues (in $\overline{\mathbb{Q}}_\ell$) of $F^*_{X,q}$ acting on $H^i_c(X, \mathbb{Q}_\ell)$ are algebraic numbers of weight less than or equal to i (in the sense of [23] Définition 1.2.1). In particular all the roots (in $\overline{\mathbb{Q}}_\ell$) of an irreducible factor of the minimal polynomial of $F^*_{X,q}$ (acting on $H^i_c(X, \mathbb{Q}_\ell)$) are algebraic numbers. Moreover $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ acts transitively on the set of these roots and hence all of them have the same weight.

For any X_0 and any integer i, the $\mathbb{Q}_\ell[t]$-module $H^i_c(X, \mathbb{Q}_\ell)$ is canonically a direct sum of $\mathbb{Q}_\ell[t]$-sub-modules of $H^i_c(X, \mathbb{Q}_\ell)$, such that each sub-module is supported only on an (ideal defined by) irreducible factor of the minimal polynomial of $F^*_{X,q}$ acting on $H^i_c(X, \mathbb{Q}_\ell)$. Thus each of these sub-modules is pure of a fixed weight. Now define $W_k H^i_c(X, \mathbb{Q}_\ell)$ to be the direct sum of those sub-modules whose weights are less than or equal to k. Hence purity of $\mathrm{Gr}^k_W H^i_c(X, \mathbb{Q}_\ell)$ is obvious from construction.

Functoriality and uniqueness are an immediate consequence of that fact that, there are no non-zero morphisms between $\mathbb{Q}_\ell[t]$ modules with disjoint supports. Strictness is also obvious from the construction of the filtration (from an underlying direct sum). □

Remark 2.3.3. Alternatively $H^i_c(X, \mathbb{Q}_\ell) = {}^pH^i(R\pi_{X!}\mathbb{Q}_{\ell,X})$, where ${}^pH^i$ is the i^{th} perverse cohomology on $D^b(\mathrm{Spec}\,(k), \mathbb{Q}_\ell)$ for the middle perversity (see [28] 2.2.18). Thus $H^i_c(X, \mathbb{Q}_\ell)$ is a mixed perverse sheaf on $\mathrm{Spec}\,(k)$ and consequently has a functorial weight filtration (see [28] Théorème 5.3.5).

Remark 2.3.4. Suppose X is a finite type and separated scheme over k (an algebraic closure of a finite field). Then after choosing a model for X over a finite sub-field of k and using Proposition 2.3.2, one can associate a weight filtration on $H^i_c(X, \mathbb{Q}_\ell)$. Further, uniqueness and functoriality (in Proposition 2.3.2) imply that, this filtration is independent of the chosen model and is functorial for proper self-maps of X/k.

Let k be either the algebraic closure of a finite field or the field of complex numbers $\mathbb{C}$. Fix a prime ℓ invertible in k (if $\mathrm{char}(k) > 0$), and an embedding

$$\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}. \tag{2.3.1}$$

Suppose X is a proper scheme over k. Let $H^*(X)$ be the ℓ-adic cohomology of X (when $\mathrm{char}(k) > 0$) with its increasing weight filtration (Proposition 2.3.2, Remark 2.3.4) or the singular cohomology $H^*(X(\mathbb{C}), \mathbb{Q})$ (when $k = \mathbb{C}$), with its Mixed Hodge structure ([25] Proposition 8.1.20). Let $W_k H^*(X)$ be the associated weight filtration.

Let $f : X \to X$ be a self map of X.

Let λ_{odd} (resp. λ_{even}) be the spectral radius (with respect to τ if $\text{char}(k) > 0$) for the action of f^* on $\oplus_{i \geqslant 0} H^{2i+1}(X)$ (resp. $\oplus_{i \geqslant 0} H^{2i}(X)$). Let k_{odd} be maximal among integer k with the property that the spectral radius for the action of f^* on $\text{Gr}_W^k H^i(X)$ is λ_{odd}, where i is an odd integer. Similarly define k_{even}.

Theorem 2.3.5. *Using the above notations, we have that*

(1) $\lambda_{even} \geqslant \lambda_{odd}$.

(2) If equality holds in (1)*, then* $k_{even} \geqslant k_{odd}$.

Proof. Suppose $k = \mathbb{C}$, then choose a model $(\mathfrak{X}, \mathfrak{F})$ of (X, f) over a finitely generated domain R over $\mathbb{Z}$ such that the structure morphism $\pi : \mathfrak{X} \to \text{Spec}\,(R)$ is proper and flat (see [21] Ch. IV Théorème 8.10.5). If necessary shrink $\text{Spec}\,(R)$ to keep ℓ invertible. Then the higher direct images $R^i \pi_* \mathbb{Q}_\ell$ are constructible sheaves on $\text{Spec}\,(R)$ (see [15] Théorème 4.6.2) . Thus shrinking $\text{Spec}\,(R)$ if necessary, we can assume that $R^i \pi_* \mathbb{Q}_\ell$ is a local system for $0 \leqslant i \leqslant 2\dim(X)$.

The morphism $\mathfrak{F}^*(\mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ induces an endomorphism Φ_i of the local system $R^i \pi_* \mathbb{Q}_\ell,\ 0 \leqslant i \leqslant 2\dim(X)$, which by functoriality of the proper base change morphism for π, induces $f^* : H^i(X, \mathbb{Q}_\ell) \to H^i(X, \mathbb{Q}_\ell)$ on the geometric generic fibre base changed to k. Since $\text{Spec}\,(R)$ has been chosen so that $R^i \pi_* \mathbb{Q}_\ell$ is a local system, the specialization map between the stalk of this local system at an algebraic geometric point over a closed point (in the sense of [15] Chapitre 1, 2.3.1) and the geometric generic point (with respect to the geometric point $\text{Spec}\,(k) \to \text{Spec}\,(R)$), is an isomorphism. Moreover this isomorphism is equivariant for the induced endomorphisms Φ_i, and respects the weight filtration (see [28] 6.2.2). Thus we are reduced to the case when k is an algebraic closure of a finite field.

Suppose $\lambda_{\text{odd}} > \lambda_{\text{even}}$. We shall obtain a contradiction.

Note that for any positive integer r, the spectral radius (with respect to τ) of f^{r*} acting on the odd and even degree cohomology are λ_{odd}^r and λ_{even}^r respectively. There exists an iterate of f which maps a connected component of X into itself. Hence, f^* acting on $H^0(X, \mathbb{Q}_\ell)$ has at least one eigenvalue of modulus 1 (with respect to any τ). Thus, $\lambda_{\text{odd}} > \lambda_{\text{even}} \geqslant 1$.

Choose a model (X_0, f_0) for the pair (X, f) over a finite field $\mathbb{F}_q$, whose algebraic closure is $\mathbb{F}$. Since $\lambda_{\text{odd}} > \lambda_{\text{even}}$, there exists an integer $r >> 0$ such that,

$$\lambda_{\text{even}}^r q^{\dim(X)} < \lambda_{\text{odd}}^r. \tag{2.3.2}$$

We fix one such r. Consider the zeta function (see Definition 1.4.1) of the pair (X_0, f_0^r). It follows from Corollary 1.4.4 that this zeta function is defined as a holomorphic function in a non-trivial neighbourhood of the origin, and has a meromorphic continuation onto the entire complex plane as a rational function. Suppose that $\frac{R(t)}{Q(t)}$ is the meromorphic continuation, where $R(t)$ and $Q(t)$ are co-prime rational polynomials in t. Moreover using the trace-determinant relation (see [22] 1.5.3), one has

$$Z(X_0, f_0^r, t) = \prod_{i=0}^{2\dim(X)} P_i(t)^{(-1)^{i+1}} \text{in } \mathbb{Q}_\ell[[t]] \subset \mathbb{C}[[t]](\text{via } \tau) \tag{2.3.3}$$

where $P_i(t) = \det\left(1 - t\,(F_{X,q} \circ f^r)^*, H^i(X, \mathbb{Q}_\ell)\right),\ 0 \leqslant i \leqslant 2\dim(X)$.

Moreover, one also has

$$Z(X_0, f_0^r, t) = \frac{R(t)}{Q(t)} \text{ in } \mathbb{Q}[[t]]. \tag{2.3.4}$$

Hence (2.3.3) and (2.3.4) imply that the complex roots of $R(t)$ and $Q(t)$ are a subset of the *inverse eigenvalues* of $(F_{X,q} \circ f^r)^*$ acting on the odd and even degree cohomology, respectively. In particular, they are non-zero. Also, note that f^* and $F_{X,q}^*$ commute and hence can be simultaneously brought to a Jordan canonical form. Hence any eigenvalue of $F_{X,q}^* \circ f^{r*}$ acting on any $H^i(X, \mathbb{Q}_\ell)$ is a product of an eigenvalue of $F_{X,q}^*$ and one of f^{r*} acting on the same $H^i(X, \mathbb{Q}_\ell)$.

Let α be any complex zero of $\prod_{0 \leqslant i \leqslant \dim(X)} P_{2i}(t) \in \mathbb{C}[t]$ (via τ in (2.3.1)). Since the i^{th} compactly supported ℓ-adic cohomology is of weight less than or equal to i ([23] Théorème 1 (3.3.1)), one has,

$$\frac{1}{|\alpha|} \leqslant \lambda_{\text{even}}^r q^{\dim(X)} < \lambda_{\text{odd}}^r. \tag{2.3.5}$$

In particular, $Q(t)$ *has no roots on the closed disc of radius* $\frac{1}{\lambda_{odd}^r}$.

Since λ_{odd}^r is the spectral radius (with respect to τ) for f^{r*} acting on the oddly graded cohomology, there exists an odd index $2i+1$ and a complex root β of $P_{2i+1}(t)$ such that,

$$|\beta| = \frac{1}{\lambda_{\text{odd}}^r q^{m(\beta)}} \leqslant \frac{1}{\lambda_{\text{odd}}^r}, \tag{2.3.6}$$

where $2m(\beta)$ is a non-negative integer (corresponding to the weight of the Frobenius). In particular, β is not a root of $\prod_{0 \leqslant i \leqslant \dim(X)} P_{2i}(t) \in \mathbb{C}[t]$. Hence β is a root of $R(t)$. Thus $R(t)$ has a root in the closed disc of radius $\frac{1}{\lambda_{\text{odd}}^r}$, while $Q(t)$ has no roots on this closed disc.

The equality between the Zeta function $Z(X_0, f_0^r, t)$ and the rational function $\frac{R(t)}{Q(t)}$ is in the ring $\mathbb{Q}[[t]]$. Since the power series corresponding to $\frac{R(t)}{Q(t)}$ is convergent in an open disc not containing any of the zeroes of $Q(t)$, *the radius of convergence of* $Z(X_0, f_0^r, t) \in \mathbb{Q}[[t]] \subset \mathbb{C}[[t]]$ *is strictly larger than* $\frac{1}{\lambda_{odd}^r}$.

It follows from Corollary 1.3.13 that the zeta function $Z(X_0, f_0^k, t)$ is of the form $\exp(G(t))$, where $G(t) \in t\mathbb{Q}[[t]]$ has non-negative coefficients. Thus Lemma 2.3.1 implies that the rational function $\frac{R(t)}{Q(t)}$ coincides with the *exponential of a holomorphic function* in the disc of convergence of the zeta function $Z(X_0, f_0^r, t)$ and in particular, in an open neighbourhood of the closed disc of radius $\frac{1}{\lambda_{\text{odd}}^r}$. Hence $R(t)$ cannot have a zero in this open neighbourhood. However (2.3.6) implies that that $R(\beta) = 0,\ |\beta| \leqslant \frac{1}{\lambda_{\text{odd}}^r}$. This is a contradiction.

Hence $\lambda_{\text{even}} \geqslant \lambda_{\text{odd}}$.

Now suppose $\lambda_{\text{even}} = \lambda_{\text{odd}}$, but $k_{\text{even}} < k_{\text{odd}}$. As before we shall obtain a contradiction.

Let μ_i be the spectral radius (with respect to τ in (2.3.1)) for the action of f^* on $H^{2i}(X, \mathbb{Q}_\ell)$ for each $0 \leqslant i \leqslant \dim(X)$. Then there exists an integer $r >> 0$ such that, for any integer $i \in [0,\ \dim(X)]$ with $\mu_i \neq \lambda_{\text{even}}$, we have

$$\mu_i^r q^{\dim(X)} < \lambda_{\text{even}}^r \leqslant \lambda_{\text{even}}^r q^{\frac{k_{\text{even}}}{2}} < \lambda_{\text{odd}}^r q^{\frac{k_{\text{odd}}}{2}}. \tag{2.3.7}$$

We fix one such r. As before Lemma 1.4.3 and Corollary 1.4.4 imply that the zeta function $Z(X_0, f_0^r, t)$ has a non-trivial radius of convergence about the origin and has a meromorphic continuation of the form $R(t)/Q(t)$ onto the entire complex plane, with $R(t)$ and $Q(t)$ being co-prime rational polynomials. Also the zeroes of $R(t)$ and $Q(t)$, are a subset of the inverse eigenvalues of $(F_{X,q} \circ f^r)^*$ acting on the odd and even degree cohomology respectively.

Since the i^{th} compactly supported ℓ-adic cohomology is of weight less than or equal to i ([23] Théorème 1 (3.3.1)), the weight filtration on $H^i(X, \mathbb{Q}_\ell)$ (Proposition 2.3.2) satisfies $W_k H^i(X, \mathbb{Q}_\ell) = H^i(X, \mathbb{Q}_\ell)$ for $k \geqslant i$.

Let

$$P_{i,k}(t) := \det(1 - t(F_{X,q} \circ f^r)^*, \mathrm{Gr}_W^k H^i(X, \mathbb{Q}_\ell)),\ 0 \leqslant k \leqslant i,\ 0 \leqslant i \leqslant 2\dim(X).$$

As before,

$$P_i(t) := \det(1 - t(F_{X,q} \circ f^r)^*, H^i(X, \mathbb{Q}_\ell)),\ 0 \leqslant i \leqslant 2\dim(X).$$

Since the weight filtration is respected by the action of $F_{X,q}^* \circ f^{r*}$ (Proposition 2.3.2), one has an equality,

$$\prod_{i=0}^{\dim(X)} \prod_{k=0}^{2i} P_{2i,k}(t) = \prod_{0 \leqslant i \leqslant \dim(X)} P_{2i}(t) \in \mathbb{C}[t] \text{ (via } \tau \text{ in (2.3.1))}.$$

Let α be any complex zero of $\prod_{0 \leqslant i \leqslant \dim(X)} P_{2i}(t) \in \mathbb{C}[t]$. Then α is a zero of $P_{2i,k}(t)$ for some integer $i \in [0, \dim(X)]$ and $0 \leqslant k \leqslant 2i$. If $\mu_i \neq \lambda_{\text{even}}$, then (2.3.7) implies,

$$\frac{1}{|\alpha|} \leqslant \mu_i^r q^{\frac{k}{2}} \leqslant \mu_i^r q^{\dim(X)} < \lambda_{\text{odd}}^r q^{\frac{k_{\text{odd}}}{2}}. \tag{2.3.8}$$

On the other hand the definition of k_{even} implies that, if $\mu_i = \lambda_{\text{even}}$, then $k \leqslant k_{\text{even}}$. Thus (2.3.7) implies that

$$\frac{1}{|\alpha|} \leqslant \lambda_{\text{even}}^{r} q^{\frac{k}{2}} \leqslant \lambda_{\text{even}}^{r} q^{\frac{k_{\text{even}}}{2}} < \lambda_{\text{odd}}^{r} q^{\frac{k_{\text{odd}}}{2}}. \tag{2.3.9}$$

Thus in any case $\frac{1}{|\alpha|} < \lambda_{\text{odd}}^{r} q^{\frac{k_{\text{odd}}}{2}}$ for any complex root α of $\prod_{0 \leqslant i \leqslant \dim(X)} P_{2i}(t)$. In particular, the same holds true for any complex root of $Q(t)$.

Moreover from the definition of k_{odd} it follows that, there exist an odd index $2i+1$ and a complex root β of $P_{2i+1,k_{\text{odd}}}(t)$ such that, $\frac{1}{|\beta|} = \lambda_{\text{odd}}^{r} q^{\frac{k_{\text{odd}}}{2}}$. Hence, β is not a root of $\prod_{0 \leqslant i \leqslant \dim(X)} P_{2i}(t)$ and thus is necessarily a root of $R(t)$. Thus $R(t)$ has a root on the closed disc of radius $\frac{1}{\lambda_{\text{odd}}^{r} q^{\frac{k_{\text{odd}}}{2}}}$, while $Q(t)$ has no roots on this closed disc. Arguing as before, we arrive at a contradiction. Hence $k_{\text{odd}} \leqslant k_{\text{even}}$.

□

Corollary 2.3.6. *Let $f : X \to X$ be a self-map of a proper scheme over an arbitrary field k. Let ℓ be a prime invertible in k and $\overline{k}$ an algebraic closure of k. Fix an embedding $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$. Then, the spectral radius (with respect to τ) for the action of $f_{\overline{k}}^*$ on the entire ℓ-adic cohomology $H^*(X_{\overline{k}}, \mathbb{Q}_\ell)$ is equal to the spectral radius for its action on $\oplus_{i \geqslant 0} H^{2i}(X_{\overline{k}}, \mathbb{Q}_\ell)$.*

2.4 Two examples

The following simple example shows that the inequality (1) in Theorem 2.3.5 need not be strict.

Example 2.4.1. Let $k = \mathbb{C}$ and $X = E \times E \times \mathbb{P}^1$ where E is any elliptic curve over $\mathbb{C}$.

Let f be the automorphism of X given by $f(x, y, z) = (2x + 3y, x + 2y, z)$. We have $f = g \times 1_{\mathbb{P}^1}$, where $g : E \to E$ is the automorphism $(x, y) \mapsto (2x + 3y, x + 2y)$. Let λ_{even}, λ_{odd}, k_{even}, k_{odd} be as defined in the previous section for the action of f^* on $H^*(X(\mathbb{C}), \mathbb{Q})$.

We have an injective ring homomorphism,

$$\operatorname{End}(E \times E) \simeq \operatorname{Mat}_{2\times 2}(\operatorname{End}(E)) \hookrightarrow \operatorname{End}\left(H_1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Z})\right),$$

induced by functoriality of singular homology (see [27], Chapter IV). Hence the minimal polynomial for the linear action induced by g on $H_1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ is $t^2 - 4t + 1$. In particular the eigenvalues of g_* on $H_1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ are $\lambda_1 := 2 + \sqrt{3}$, $\lambda_2 := 2 - \sqrt{3} = \frac{1}{\lambda_1}$ with some multiplicities. By functoriality of the universal coefficient isomorphism, the eigenvalues of g^* on $H^1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ are also λ_1 and λ_2 with the same multiplicity.

The cohomology ring $H^*(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ is isomorphic (as graded rings) to the exterior algebra on the $\mathbb{Q}$-vector space $H^1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$. Further this isomorphism is equivariant for the action of g^* and since, g^* acts as identity on $H^4(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$, the mutiplicities of λ_1 and λ_2 on $H^1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ should be precisely 2 each.

Since $H^1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ is a 4-dimensional vector space over $\mathbb{Q}$, the eigenvalues of g^* on $H^2(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q}) \simeq \Lambda^2 H^1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ are $\lambda_1^2,\ \lambda_2^2,\ \lambda_1\lambda_2 = 1$ with multiplicities $1,\ 1$ and 4 respectively. On the other hand, as representations of the cyclic group $\langle g^{k*},\ k \in \mathbb{Z}\rangle$, $H^3(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$ is dual to $H^1(E(\mathbb{C}) \times E(\mathbb{C}), \mathbb{Q})$. Thus it follows from the Kunneth formula that $\lambda_{\text{even}} = \lambda_{\text{odd}} = \lambda_1^2$, while $k_{\text{even}} = 4$ and $k_{\text{odd}} = 3$.

The following example shows that Theorem 2.3.5 is false without the properness hypothesis, even for smooth varieties.

Example 2.4.2. Let T/k be a rank 2 torus (split over $\mathbb{F}_q$) and $f : T \to T$ any group automorphism of T. Then, $\ker(f \circ F^n_{T,q} - 1_T)$ is a finite étale group scheme. Since it is a sub-group scheme of a torus, its order (or rank) is co-prime to p. For any integer $n \in \mathbb{N}$, let $\mathrm{Fix}(f \circ F^n_{T,q})$ be the number of fixed points of $f \circ F^n_{T,q}$ acting on T. Then

$$\mathrm{Fix}(f \circ F^n_{T,q}) = |P_f(q^n)|$$

where $P_f(t) := \det(1 - tM_f; X^*(T))$, and M_f is the linear map on the co-character lattice $X^*(T)$, induced by f.

The only non-trivial compactly supported ℓ-adic cohomology groups of T are $H^2_c(T, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell$, $H^3_c(T, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-1)^{\oplus 2}$ and $H^4_c(T, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell(-2)$. Thus for all integers $n \geqslant 1$,

$$\mathrm{Tr}((f \circ F^n_{T,q})^*, H^*_c(T, \mathbb{Q}_\ell)) = q^{2n} - \mathrm{Tr}(f^*, H^3_c(T, \mathbb{Q}_\ell))q^n + \mathrm{Tr}(f^*, H^2_c(T, \mathbb{Q}_\ell)).$$

It follows from Corollary 1.3.13 that for n large enough,

$$\mathrm{Tr}((f \circ F^n_{T,q})^*, H^*_c(T, \mathbb{Q}_\ell)) = \mathrm{Fix}(f \circ F^n_{T,q}) = P_f(q^n) = q^{2n} - \mathrm{Tr}(M_f)q^n + \det(M_f).$$

Thus

$$\mathrm{Tr}((f \circ F^n_{T,q})^*, H^*_c(T, \mathbb{Q}_\ell)) = P_f(q^n)$$

for all integers $n \geqslant 1$.

Let f be chosen such that $|\mathrm{Tr}(M_f)| > 2$ and $\det(M_f) = 1$. Note that the eigenvalues of f^* acting on the compactly supported ℓ-adic cohomology are algebraic integers independent of ℓ. Since $\mathrm{Tr}((f \circ F^n_{T,q})^*, H^*_c(T, \mathbb{Q}_\ell)) = P_f(q^n)$, it follows that, for any embedding $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$,

(1) f^* acting on $H^3_c(T, \mathbb{Q}_\ell)$ has at least one eigenvalue of modulus greater than 1.

(2) f^* acts as identity on $H^2_c(T, \mathbb{Q}_\ell)$ and $H^4_c(T, \mathbb{Q}_\ell)$.

Hence $\lambda_{\text{odd}} > \lambda_{\text{even}}$.

A careful look at the proof of Theorem 2.3.5 shows that the failure of Theorem 2.3.5 without the properness hypothesis (as in Example 2.4.2), can be attributed to $n_0(f^k)$ (see Definition 1.3.14) being strictly greater than 1. This motivates us to consider the growth of an upper bound for $n_0(f^k)$ with respect to k for non-proper varieties. This will be a topic of interest in the next chapter.

Chapter 3

Growth of an upper bound for $n_0(f^k)$ with respect to k

In this section, we study the growth of an upper bound for $n_0(f^k)$ (see Definition 1.3.14) with respect to k. A crucial input for us is Varshavsky's notion of a 'contracting' correspondence (see [8] 2.1). We recall the necessary notations and definitions from [8].

We continue using the notations and conventions from previous sections.

3.1 Varshavsky's trace formula

We work over a separably closed base field k. All schemes are assumed to be separated and finite type over k.

Let $c : X \to Y$ be a morphism of schemes.

Definition 3.1.1 (Ramification along a closed sub-scheme)**.** For a reduced closed subscheme $Z \subset Y$ its *ramification along* c is the smallest $n \in \mathbb{N}$ such that $c^{-1}(Z) \subseteq (c^{-1}(Z)_{red})_n$. We denote this by $\mathrm{Ram}(Z, c)$.

Definition 3.1.2 (Ramification degree of a morphism)**.** If c above is quasi-finite, then $\mathrm{ram}(c)$ is defined as the maximum of $\mathrm{ram}(y, c)$, as y varies over all the closed points of Y. We denote this by $\mathrm{ram}(c)$.

Remark 3.1.3. Note that our notation for ramification along a closed subscheme differs from the one in [8] to avoid any possibility of a confusion with the notation for the ramification degree of a morphism.

Now let $c : C \to X \times_k X$ be a self-correspondence of X.

Definition 3.1.4 (Invariant closed subset)**.** A closed subset $Z \subseteq X$ is said to be c-*invariant* if $c_1(c_2^{-1}(Z))$ is set theoretically contained in Z.

Definition 3.1.5 (Locally invariant closed sub-set)**.** A closed subset $Z \subseteq X$ is said to be *locally c-invariant* if for each $x \in Z$ there exists an open neighbourhood U of x in X such that, $Z \cap U$ is $[c]|_U$-invariant (see Definition 1.1.8) .

Example 3.1.6. If c_2 is quasi-finite then any closed point of X is locally c-invariant (see [8] Example 1.5.2).

Remark 3.1.7. Suppose $[c]$ is a correspondence, locally invariant along a closed subset Z. Let u be a cohomological correspondence from $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ to itself lifting $[c]$ (Definition 1.1.9). Then we can define a self-correspondence $[c]|_Z$ of Z and a cohomological self-correspondence $u|_Z$ of $\mathcal{F}|_Z$ lifting $[c]|_Z$ as in [8], 1.5.6.

Definition 3.1.8. A closed subscheme Z is said to be *stabilized* by c if $c_2^{-1}(Z)$ is a closed subscheme of $c_1^{-1}(Z)$.

Recall that for a closed subscheme Z of X, Z_k is the closed subscheme of X defined by the ideal $\mathcal{I}_Z^k$.

Definition 3.1.9. c is said to be *contracting near a closed subscheme* $Z \subseteq X$ if c stabilizes Z and $c_2^{-1}(Z_{n+1})$ is a closed subscheme of $c_1^{-1}(Z_n)$ for some $n \geqslant 1$.

Definition 3.1.10. A closed subscheme $Z \subseteq X$ is said to be *contracting in a neighbourhood of fixed points* if there exists an open subscheme W of C containing $\mathrm{Fix}(c)$ such that $[c]|_W$ is contracting near Z (see Definition 1.1.8).

Unless otherwise stated, henceforth in this section, we work over k, an algebraic closure of a finite field $\mathbb{F}_q$.

We record the following obvious lemma.

Lemma 3.1.11. *Let $[c] := (C, c_1, c_2)$ be a self correspondence of X defined over $\mathbb{F}_q$. Let $[\bar{c}] := (\overline{C}, \bar{c}_1, \bar{c}_2)$ be a proper self correspondence of $\overline{X}$ also defined over $\mathbb{F}_q$. Suppose there exists an open immersion $[j] = (j_1, j^{\#}, j_2) : [c] \to [\bar{c}]$ defined over $\mathbb{F}_q$. Then, for all $n \geqslant 1$, $[j] := (j^{\#}, j_1, j_2) : [c]^{(n)} \to [\bar{c}]^{(n)}$ (see Definition 1.3.10) is an open immersion, which is a compactification, if $[j] : [c] \to [\bar{c}]$ is a compactification.*

An important result in [8] is the following trace formula.

Theorem 3.1.12. *([8], Theorem 2.3.2)*

Let $c : C \to X \times_k X$ be a correspondence defined over $\mathbb{F}_q$.

(1) Suppose c_2 is quasi-finite. Then for any $n \in \mathbb{N}$ with $q^n > ram(c_2)$, the scheme $Fix(c^{(n)})$ is zero-dimensional.

(2) Let $U \subseteq X$ be an open subscheme also defined over $\mathbb{F}_q$ such that $c_1|_{c_1^{-1}(U)}$ is proper, $c_2|_{c_2^{-1}(U)}$ is quasi-finite, and the closed subset $X \backslash U$ is locally c-invariant.

Then there exists a positive integer $d \geqslant ram(c_2|_{c_2^{-1}(U)})$ with the following property: for every $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ with $\mathcal{F}|_{X\backslash U} = 0$ and every $n \in \mathbb{N}$ with $q^n > d$, and for any cohomological self-correspondence u of $\mathcal{F}$ lifting $[c]^{(n)}$, one has

$$Tr(R\Gamma_c(u)) = \sum_{y \in Fix(c^{(n)}) \cap c_2^{-1}(U)} Tr(u_y).$$

Here u_y is the induced endomorphism on $\mathcal{F}_y$ (see Example 1.2.1).

(3) In the notation of (2) as above, assume that X and C are proper over k. Then

$$d =: max\{ram(c_2|_{c_2^{-1}(U)}), ram(c_2, X \backslash U)\}$$

satisfies the conclusion of (2).

Remark 3.1.13. When $U = X$ the conclusion of assertion (2) above is the same as that of Fujiwara ([6] Corollary 5.4.5).

3.1.1 A compactification of the transpose of a graph

Let X be a finite type and separated scheme over k and $f : X \to X$ a proper morphism. Assume that the pair (X, f) is defined over $\mathbb{F}_q$. Let $[\Gamma_f^t]$ be the associated self correspondence of X as in Example 1.1.3. Let $j : X \hookrightarrow \overline{X}$ be an arbitrary compactification of X (assumed to be defined over $\mathbb{F}_q$). Let $\partial\overline{X} := \overline{X} \backslash X$, with the reduced induced structure.

Let $\overline{\Gamma_f^t}$ be the Zariski closure of Γ_f^t inside $\overline{X} \times_k \overline{X}$. Let $\overline{c}_{1,f}$ and $\overline{c}_{2,f}$ be the map induced from $\overline{\Gamma_f^t}$ to $\overline{X}$, by the first and second projection respectively. Let $j_f : X \hookrightarrow \overline{\Gamma_f^t}$ be the open immersion. Let $c_{1,f}$ and $c_{2,f}$ be the restrictions of $\overline{c}_{1,f}$ and $\overline{c}_{2,f}$ respectively to X (via j_f). Let $\partial\overline{\Gamma}_f^t$ be the complement of X inside $\overline{\Gamma}_f^t$ with the reduced induced structure. For $i = 1, 2$ let $\partial\overline{c}_{i,f}$ be the restriction of $\overline{c}_{i,f}$ to $\partial\overline{\Gamma}_f^t$. Then $[\overline{f}] := (\overline{\Gamma}_f^t, \overline{c}_{1,f}, \overline{c}_{2,f})$ (a self correspondence of $\overline{X}$) is a compactification of $[\Gamma_f^t]$ via $[j_f] := (j, j_f, j)$.

Since f is proper and $\overline{[f]} := (\overline{\Gamma_f^t}, \overline{c}_{1,f}, \overline{c}_{2,f})$ is a compactification of $[\Gamma_f^t]$, Lemma 1.1.7 implies that $\partial\overline{X}$ is $\overline{[f]}$-invariant. Hence we have a commutative diagram,

$$\begin{tikzcd}
X \arrow[r, hook, "j"] & \overline{X} & \partial\overline{X} \arrow[l, hook', "i"'] \\
X \arrow[u, "c_{1,f}=f"] \arrow[d, "c_{2,f}=1_X"'] \arrow[r, hook, "j_f"] & \overline{\Gamma}_f^t \arrow[u, "\overline{c}_{1,f}"'] \arrow[d, "\overline{c}_{2,f}"'] & \partial\overline{\Gamma}_f^t \arrow[l, hook', "i_f"'] \arrow[u, "\partial\overline{c}_{1,f}"'] \arrow[d, "\partial\overline{c}_{2,f}"'] \\
X \arrow[r, hook, "j"] & \overline{X} & \partial\overline{X} \arrow[l, hook', "i"']
\end{tikzcd}$$

Corollary 3.1.14. *For all $n > \log_q(Ram(\partial\overline{X}, \overline{c}_{2,f}))$, one has*

$$Fix(f \circ F_{X,q}^n) = Tr((f \circ F_{X,q}^n)^*, H_c^*(X, \mathbb{Q}_\ell)).$$

Here $Fix(f \circ F^n_{X,q})$ is the number of fixed points of $f \circ F^n_{X,q}$(see Lemma 1.3.6).

Proof. For any $n > \log_q(\mathrm{Ram}(\partial\overline{X}, \overline{c}_{2,f}))$ let $u_0^{(n)}$ be the cohomological self correspondence of $\mathbb{Q}_\ell$ lifting $[\Gamma^t_{F^n_{X,q}\circ f}]$ as in Example 1.1.10. It follows from Lemma 3.1.11 that $[\overline{f}]^{(n)}$ is a compactification of $[\Gamma^t_{F^n_{X,q}\circ f}]$. Let $u^{(n)}$ be the cohomological self correspondence of $j_!\mathbb{Q}_\ell$ obtained as in Example 1.2.8 lifting $[\overline{f}]^{(n)}$. Example 1.2.8 implies that the endomorphisms $R\Gamma_c(u^{(n)})$ and $R\Gamma_c(u_0^{(n)})$ of $R\Gamma_c(X, \mathbb{Q}_\ell)$ are identical. The result then follows by applying part (3) of Theorem 3.1.12 to $u^{(n)}$ (see Example 1.2.1).

$\square$

Corollary 3.1.14 implies that $n_0(f) \leqslant \mathrm{Ram}(\partial\overline{X}, \overline{c}_{2,f})$ (see Definition 1.3.14). The procedure above gives an 'effective' upper bound for $n_0(f)$ in terms of the geometry of the chosen compactification of X. However there is no obvious relationship between these bounds for $n_0(f)$ and $n_0(f^r)$, $r > 1$.

Given the Lefschetz-Verdier trace formula, one obtains a trace formula as in Varshavsky [8], using additivity of trace maps (see [8] Section 5) and making the correspondence contract along the boundary. The contraction along the boundary ensures that the contribution to the local terms of the trace formula coming from the boundary is trivial. The condition (3) in Theorem 3.1.12 ensures contraction along the boundary after twisting the correspondence with a high enough iterate of the Frobenius. The key step is the following theorem.

Theorem 3.1.15. *([8], Theorem 2.1.3)*

Let $c : C \to X \times_k X$ be a correspondence contracting near a closed subscheme $Z \subseteq X$ in a neighbourhood of fixed points, and let β be an open connected subset of $Fix(c)$ such that $c'(\beta) \cap Z \neq \varnothing$ (see 1.2.1). Then

(1) β is contained set-theoretically in $c'^{-1}(Z)$, hence β is an open connected subset of $Fix(c|_Z)$ (see Remark 3.1.7).

(2) For every cohomological correspondence u from $\mathcal{F}$ to itself lifting c, one has $\mathcal{T}r_\beta(u) = \mathcal{T}r_\beta(u|_Z)$. In particular, if β is proper over k, then $LT_\beta(u) = LT_\beta(u|_Z)$ (see Remark 3.1.7).

Remark 3.1.16. The theorem above holds true over arbitrary separably closed fields and c need *not* be proper.

The following corollary can be read off from the proof of Theorem 3.1.12 in [8].

Corollary 3.1.17. *Let $c : C \to X \times_k X$ be a self-correspondence of a scheme X. Let $U \subseteq X$ be an open subscheme and $\mathcal{F} \in D^b_{ctf}(X, \Lambda)$ be supported on U. Suppose also that c is contracting near the closed subscheme Z, where $Z_{red} = X \backslash U$. Let u be a cohomological self-correspondence of $\mathcal{F}$ lifting $[c]$. Then*

$$\sum_{\beta\in\pi_0(Fix(c))} LT_\beta(u) = \sum_{\beta\in(\pi_0(Fix(c|_U)))} LT_\beta(u|_U)$$

Remark 3.1.18. That the right hand side makes sense is a part of the corollary (see Chapter 1, Section 1.1.1 for the definition of $u|_U$).

Proof. Consider the following diagram of schemes (see 1.2.1),

$$\begin{tikzcd} \mathrm{Fix}(c) \arrow[r, "c'"] \arrow[d, "\Delta'"] & X \arrow[d, "\Delta"] \\ C \arrow[r, "c"] & X \times X \end{tikzcd}$$

If a connected component β of $\mathrm{Fix}(c)$ is such that $c'(\beta)\cap Z = \varnothing$ then, $c_1(\beta) = c_2(\beta)$ is disjoint from Z. Hence, one has the inclusion $\beta \subset c_1^{-1}(U)\cap c_2^{-1}(U)$. Thus β is a connected component of $\mathrm{Fix}(c|_U)$. On the other hand if $c'(\beta)\cap Z \neq \varnothing$, Theorem 3.1.15 implies that $LT_\beta(u) = LT_\beta(u|_Z)$. Since $\mathcal{F}$ restricted to Z is trivial and $u|_Z$ is a cohomological self-correspondence of $\mathcal{F}|_Z$ (see Remark 3.1.7), one necessarily has $LT_\beta(u) = 0$ if $c'(\beta)\cap Z \neq \varnothing$.

Moreover (1) of Theorem 3.1.15 implies that if $\beta \in \pi_0(\mathrm{Fix}(c|_U))$ is such that $\beta \subset \beta' \in \pi_0(\mathrm{Fix}(c))$ then, $c'(\beta')\cap Z = \varnothing$. Hence $\beta = \beta'$. Thus the connected components of $\mathrm{Fix}(c|_U)$ are precisely the connected components of $\mathrm{Fix}(c)$ whose image under c' does not intersect Z. In particular these connected components are proper over k. Hence the right hand side makes sense. Since the local term at a connected component $\beta \subseteq \mathrm{Fix}(c)$ depends only on an open neighbourhood of β, the result follows.

□

3.2 Compactifications adapted to iteration and contraction

Let X be a finite type and separated scheme over a field k, an algebraic closure of a finite field $\mathbb{F}_q$. Let $\overline{X}$ be a compactification of X. Let $j : X \hookrightarrow \overline{X}$ be the open dense inclusion. Let $\partial\overline{X}$ be the complement of X in $\overline{X}$, with the reduced induced structure. Let $i : \partial\overline{X} \hookrightarrow \overline{X}$ be the closed immersion. Assume that X, $\overline{X}$ and j are defined over $\mathbb{F}_q$.

Let f and g be two proper self maps of X both defined over $\mathbb{F}_q$. Let $[\overline{f}] = (\overline{C}_f, \overline{c}_{1,f}, \overline{c}_{2,f})$ and $[\overline{g}] := (\overline{C}_g, \overline{c}_{1,g}, \overline{c}_{2,g})$ be self-correspondences of $\overline{X}$ compactifying $[\Gamma_f^t]$ and $[\Gamma_g^t]$ respectively, and also defined over $\mathbb{F}_q$. Let $j_f : X \hookrightarrow \overline{C}_f$ and $j_g : X \hookrightarrow \overline{C}_g$ be the open dense inclusions. Also let $\partial\overline{C}_f$ be the complement of X in $\overline{C}_f$ with the reduced induced structure. Similarly define $\partial\overline{C}_g$. Let $i_f : \partial\overline{C}_f \hookrightarrow \overline{C}_f$ be the closed immersion and similarly one also has i_g. Let $c_{1,f}$ and $c_{2,f}$ be the restrictions of $\overline{c}_{1,f}$ and $\overline{c}_{2,f}$ respectively to $\Gamma_f^t = X \hookrightarrow \overline{C}_f$. Similarly define $c_{1,g}$ and $c_{2,g}$. Let $\partial\overline{c}_{i,f},\ i = 1, 2$ be the restriction of $\overline{c}_{i,f},\ i = 1, 2$ to $\partial\overline{C}_f$. Similarly define $\partial\overline{c}_{i,g},\ i = 1, 2$.

Since f and g are proper, Lemma 1.1.7 implies that the dense open immersions $X \hookrightarrow \overline{c}_{1,f}^{-1}(X)$ and $X \hookrightarrow \overline{c}_{1,g}^{-1}(X)$ are isomorphisms. Thus $\partial\overline{C}_f \subseteq \overline{c}_{i,f}^{-1}(\partial\overline{X}),\ i = 1, 2$ and $\partial\overline{C}_g \subseteq \overline{c}_{i,g}^{-1}(\partial\overline{X}),\ i = 1, 2$. Moreover, the supports of $\partial\overline{C}_f$ and $\overline{c}_{i,f}^{-1}(\partial\overline{X}),\ i = 1, 2$ are equal and a similar result is true for g.

Hence as before one has the following commutative diagram,

$$\begin{tikzcd}
X \arrow[r, hook, "j"] & \overline{X} & \partial\overline{X} \arrow[l, hook', "i"'] \\
X \arrow[u, "c_{1,f}=f"] \arrow[d, "c_{2,f}=1_X"'] \arrow[r, hook, "j_f"] & \overline{C}_f \arrow[u, "\overline{c}_{1,f}"] \arrow[d, "\overline{c}_{2,f}"] & \partial\overline{C}_f \arrow[l, hook', "i_f"'] \arrow[u, "\partial\overline{c}_{1,f}"] \arrow[d, "\partial\overline{c}_{2,f}"] \\
X \arrow[r, hook, "j"] & \overline{X} & \partial\overline{X} \arrow[l, hook', "i"']
\end{tikzcd}.$$

There exists a similar diagram for g too.

Proposition 3.2.1. *Let $[\overline{f}]$ and $[\overline{g}]$ be compactifications of $[\Gamma_f^t]$ and $[\Gamma_g^t]$ as above. Suppose that $\partial\overline{X}$ is $[\overline{f}]^{(n)}$ and $[\overline{g}]^{(m)}$ contracting (see Definition 3.1.9). Then, there exists a self-correspondence $[\widetilde{g \circ f}]$ of $\overline{X}$ and a morphism $[\widetilde{j}_{g\circ f}] := (\widetilde{j}_{g\circ f,1}, \widetilde{j}_{g\circ f}^{\#}, \widetilde{j}_{g\circ f,2})$ from $[\Gamma_{g\circ f}^t]$ to $[\widetilde{g \circ f}]$ such that,*

1. *$[\widetilde{g \circ f}]$ is a self-correspondence of $\overline{X}$, also defined over $\mathbb{F}_q$.*
2. *$[\widetilde{g \circ f}]$ is a compactification of $\Gamma_{g\circ f}^t$ (via $[\widetilde{j}_{g\circ f}]$) and $\widetilde{j}_{g\circ f,1} = \widetilde{j}_{g\circ f,2} = j$.*
3. *$\partial\overline{X}$ is $[\widetilde{g \circ f}]^{(m+n)}$-contracting.*

Proof. Consider the following diagram,

$$\begin{tikzcd}
 & & X \arrow[r, "g"] \arrow[d, hook, "j_g"] & X \arrow[d, hook, "j"] \\
 & \overline{C}_f \times_{\overline{X}} \overline{C}_g \arrow[r, "\overline{c}'_{1,f}"] \arrow[d, "\overline{c}'_{2,g}"] & \overline{C}_g \arrow[r, "\overline{c}_{1,g}"] \arrow[d, "\overline{c}_{2,g}"] & \overline{X} \\
X \arrow[r, hook, "j_f"] \arrow[d, "c_{2,f}=1_X"] & \overline{C}_f \arrow[r, "\overline{c}_{1,f}"] \arrow[d, "\overline{c}_{2,f}"] & \overline{X} & \\
X \arrow[r, hook, "j"] & \overline{X} & &
\end{tikzcd} \tag{3.2.1}$$

Each square in the above diagram is cartesian.

Here $\overline{c}'_{1,f}$ and $\overline{c}'_{2,g}$ arise from base change of $\overline{c}_{1,f}$ and $\overline{c}_{2,g}$ respectively. Let $[\widetilde{g \circ f}] := (\overline{C}_f \times_{\overline{X}} \overline{C}_g, \overline{c}_{1,g} \circ \overline{c}'_{1,f}, \overline{c}_{2,f} \circ \overline{c}'_{2,g})$.

Note that the composite morphism $\overline{c}_{1,f} \circ j_f$ factors via $j : X \hookrightarrow \overline{X}$ and hence, the base change of $j_f : X \hookrightarrow \overline{C}_f$ by $\overline{c}'_{2,g}$, is at once isomorphic to X under the projection map and also an dense open immersion inside $\overline{C}_f \times_{\overline{X}} \overline{C}_g$.

Hence there exists a dense open immersion $\widetilde{j}^{\#}_{g\circ f}$ from X to $\overline{C}_f \times_{\overline{X}} \overline{C}_g$ such that $[\widetilde{j}_{g\circ f}] := (j, \widetilde{j}^{\#}_{g\circ f}, j)$ from $\Gamma^t_{g\circ f}$ to $[\widetilde{g \circ f}]$ is a compactification. Further it is clear that $[\widetilde{g \circ f}]$ is also defined over $\mathbb{F}_q$.

Let $\partial(\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g)$ be the complement in $\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g$ of X with the reduced induced structure.

Since, $\partial\overline{X}$ is $[f]^{(n)}$ and $[g]^{(m)}$ contracting, it is in particular stabilized by $[f]^n$ and $[g]^m$. Thus, there exist inclusions of closed subschemes

$$\overline{c}^{-1}_{2,f}(\partial\overline{X}) \subseteq \left(F^n_{\overline{X},q} \circ \overline{c}_{1,f}\right)^{-1} \left(\partial\overline{X}\right) \tag{3.2.2}$$

and

$$\overline{c}^{-1}_{2,g}(\partial\overline{X}) \subseteq \left(F^m_{\overline{X},q} \circ \overline{c}_{1,g}\right)^{-1} \left(\partial\overline{X}\right). \tag{3.2.3}$$

Hence (3.2.2) implies that

$$(\overline{c}_{2,f} \circ \overline{c}'_{2,g})^{-1}(\partial\overline{X}) \subseteq \overline{c}'^{-1}_{2,g} \left(\left(F^n_{\overline{X},q} \circ \overline{c}_{1,f}\right)^{-1} \left(\partial\overline{X}\right)\right). \tag{3.2.4}$$

Since $\overline{c}_{2,g} \circ \overline{c}'_{1,f} = \overline{c}_{1,f} \circ \overline{c}'_{2,g}$ (see Diagram 3.2.1) , one has

$$\overline{c}'^{-1}_{2,g} \left(\left(F^n_{\overline{X},q} \circ \overline{c}_{1,f}\right)^{-1} \left(\partial\overline{X}\right)\right) = \overline{c}'^{-1}_{1,f} \left(\left(F^n_{\overline{X},q} \circ \overline{c}_{2,g}\right)^{-1} \left(\partial\overline{X}\right)\right). \tag{3.2.5}$$

Moreover, since $F^n_{\overline{X},q} \circ \overline{c}_{2,g} = \overline{c}_{2,g} \circ F^n_{\overline{C}_g,q}$, (3.2.3) implies that

$$\overline{c}'^{-1}_{1,f} \left(\left(F^n_{\overline{X},q} \circ \overline{c}_{2,g}\right)^{-1} \left(\partial\overline{X}\right)\right) \subseteq \overline{c}'^{-1}_{1,f} \left(\left(F^{m+n}_{\overline{X},q} \circ \overline{c}_{1,g}\right)^{-1} \left(\partial\overline{X}\right)\right). \tag{3.2.6}$$

Combining (3.2.4), (3.2.5) and (3.2.6) one gets,

$$\left(\overline{c}_{2,f} \circ \overline{c}'_{2,g}\right)^{-1} \left(\partial\overline{X}\right) \subseteq \left(F^{m+n}_{\overline{X},q} \circ \left(\overline{c}_{1,g} \circ \overline{c}'_{1,f}\right)\right)^{-1} \left(\partial\overline{X}\right).$$

Thus, $\partial\overline{X}$ is stabilized by $[\widetilde{g \circ f}]^{(m+n)}$.

Moreover by the contracting properties of $[\overline{f}]^{(n)}$ and $[\overline{g}]^{(m)}$, there exist $d, e \in \mathbb{N}$ and inclusions of closed subschemes,

$$\overline{c}^{-1}_{2,f} \left(\left(\partial\overline{X}\right)_{d+1}\right) \subseteq \left(F^n_{\overline{X},q} \circ \overline{c}_{1,f}\right)^{-1} \left(\left(\partial\overline{X}\right)_d\right) \tag{3.2.7}$$

and

$$\overline{c}_{2,g}^{-1}\left(\left(\partial\overline{X}\right)_{e+1}\right) \subseteq \left(F^{m}_{\overline{X},q}\circ \overline{c}_{1,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{e}\right). \tag{3.2.8}$$

Hence (3.2.7) implies that

$$\left(\overline{c}_{2,f}\circ\overline{c}'_{2,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{de+d+e+1}\right)\subseteq \overline{c}'^{-1}_{2,g}\left(\left(F^{n}_{\overline{X},q}\circ\overline{c}_{1,f}\right)^{-1}\left(\left(\partial\overline{X}\right)_{d(e+1)}\right)\right). \tag{3.2.9}$$

Since $\overline{c}_{2,g}\circ\overline{c}'_{1,f}=\overline{c}_{1,f}\circ\overline{c}'_{2,g}$ (see Diagram 3.2.1), one has

$$\overline{c}'^{-1}_{2,g}\left(\left(F^{n}_{\overline{X},q}\circ\overline{c}_{1,f}\right)^{-1}\left(\left(\partial\overline{X}\right)_{d(e+1)}\right)\right)=\overline{c}'^{-1}_{1,f}\left(\left(F^{n}_{\overline{X},q}\circ\overline{c}_{2,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{d(e+1)}\right)\right). \tag{3.2.10}$$

Moreover, since $F^{n}_{\overline{X},q}\circ\overline{c}_{2,g}=\overline{c}_{2,g}\circ F^{n}_{\overline{C}_g,q}$, (3.2.8) implies that

$$\overline{c}'^{-1}_{1,f}\left(\left(F^{n}_{\overline{X},q}\circ\overline{c}_{2,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{d(e+1)}\right)\right)\subseteq\overline{c}'^{-1}_{1,f}\left(\left(F^{m+n}_{\overline{X},q}\circ\overline{c}_{1,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{de}\right)\right). \tag{3.2.11}$$

Thus, combining (3.2.9), (3.2.10) and (3.2.11) one has,

$$\left(\overline{c}_{2,f}\circ\overline{c}'_{2,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{de+d+e+1}\right)\subseteq\left(F^{m+n}_{\overline{X},q}\circ\left(\overline{c}_{1,g}\circ\overline{c}'_{1,f}\right)\right)^{-1}\left(\partial\overline{X}\right)_{de}.$$

Since

$$\left(\overline{c}_{2,f}\circ\overline{c}'_{2,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{de+1}\right)\subseteq\left(\overline{c}_{2,f}\circ\overline{c}'_{2,g}\right)^{-1}\left(\left(\partial\overline{X}\right)_{de+d+e+1}\right),$$

we conclude that $\partial\overline{X}$ is $[\widetilde{g\circ f}]^{(m+n)}$-contracting.

□

Remark 3.2.2. Suppose $[\overline{f}]$ and $[\overline{g}]$ above, are the compactifications of $[\Gamma^t_f]$ and $[\Gamma^t_g]$ as obtained in 3.1.1. Let $\widetilde{g\circ f}$ denote the morphism induced from $\overline{\Gamma}^t_f\times_{\overline{X}}\overline{\Gamma}^t_g$ to $\overline{X}\times_k\overline{X}$ by $[\widetilde{g\circ f}]$. Denote by $\overline{\Gamma}^t_{g\circ f}$, the Zariski closure of $\Gamma^t_{g\circ f}$ inside $\overline{X}\times_k\overline{X}$. Let $j_{g\circ f}$ denote the inclusion of X inside $\overline{\Gamma}^t_{g\circ f}$. Then one has a commutative diagram,

$$\begin{tikzcd}
X \arrow[r, hook, "\tilde{j}^{\#}_{g\circ f}"] \arrow[d, hook, "j_{g\circ f}"] & \overline{\Gamma}^t_f\times_{\overline{X}}\overline{\Gamma}^t_g \arrow[d, "\widetilde{g\circ f}"] \\
\overline{\Gamma}^t_{g\circ f} \arrow[r, hook] & \overline{X}\times_k\overline{X}
\end{tikzcd}$$

Clearly $\overline{\Gamma}^t_{g\circ f} \subset \widetilde{g\circ f}(\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g)$. If $\overline{\Gamma}^t_{g\circ f}$ is properly contained in $\widetilde{g\circ f}(\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g)$, choose a point $x \in (\overline{\Gamma}^t_{g\circ f})^c \cap \widetilde{g\circ f}(\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g)$. In particular $x \notin X$ and hence x is in the image of an element $x' \in \partial(\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g)$. Hence $(\widetilde{g\circ f})^{-1}((\overline{\Gamma}^t_{g\circ f})^c \cap \widetilde{g\circ f}(\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g))$ is a non-empty open set containing x' and hence intersects $X \hookrightarrow \overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g$ non trivially, a contradiction.

Hence there is a proper birational morphism from $\overline{\Gamma}^t_f \times_{\overline{X}} \overline{\Gamma}^t_g$ to $\overline{\Gamma}^t_{g\circ f}$, the more 'obvious' choice for a compactification of $\Gamma^t_{g\circ f}$. However this morphism being in the 'wrong' direction does not help in proving that $[\overline{g\circ f}]$ has the right contraction properties.

Remark 3.2.3. Note that by construction $[\widetilde{g\circ f}]$ is independent of m or n. Hence, if $\partial\overline{X}$ is contracted by $[\overline{f}]^{(n')}$ and $[\overline{g}]^{(m')}$ for some $m', n' \in \mathbb{N}$, then it is contracted by $[\widetilde{g\circ f}]^{(m'+n')}$.

Remark 3.2.4. As pointed out by the referee, an argument along the lines of the proof of Proposition 3.2.1 shows that $[\widetilde{g\circ f}]^{(m+n)}$ is contracting along $\partial\overline{X}$ even when $\partial\overline{X}$ is *only stabilized* by $[g]^{(m)}$.

3.3 A trace formula

Recall that X was a finite type and separated scheme over k, defined over a finite field $\mathbb{F}_q$, and $f : X \to X$ was a proper self map, also defined over $\mathbb{F}_q$.

Theorem 3.3.1. *There exists an integer $N(f) \geqslant 1$ such that for all integers $n \geqslant N(f)$ and $k \geqslant 1$,*

$$Fix(f^k \circ F^{nk}_{X,q}) = Tr((f^k \circ F^{nk}_{X,q})^*, H^*_c(X, \mathbb{Q}_\ell)), \tag{3.3.1}$$

where $Fix(f^k \circ F^{nk}_{X,q})$ is the number of fixed points of $f^k \circ F^{nk}_{X,q}$ acting on X. Moreover when X is proper we can take $N(f) = 1$.

Proof. When X is proper the claim follows from Corollary 1.3.13. Hence we can assume that X is not proper.

Let $[\overline{f}]$ be the compactification of Γ^t_f as defined in 3.1.1. As observed in 3.1.1, the closed sub-schemes $\overline{c}^{-1}_{1,f}(\partial\overline{X})$ and $\overline{c}^{-1}_{2,f}(\partial\overline{X})$ have the same support. Hence

$$\overline{c}^{-1}_{2,f}(\partial\overline{X}) \subseteq \left(\overline{c}^{-1}_{2,f}\left(\partial\overline{X}\right)\right)_{red,r} = \left(\overline{c}^{-1}_{1,f}\left(\partial\overline{X}\right)\right)_{red,r} \subseteq \overline{c}^{-1}_{1,f}\left(\left(\partial\overline{X}\right)_r\right),$$

where $r = \mathrm{Ram}(\partial\overline{X}, \overline{c}_{2,f})$ (see Definition 3.1.1).

In particular, since $\partial\overline{X}$ is also defined over $\mathbb{F}_q$,

$$\overline{c}^{-1}_{2,f}\left(\partial\overline{X}\right) \subseteq \overline{c}^{-1}_{1,f}\left(\left(\partial\overline{X}\right)_r\right) \subseteq \left(F^n_{\overline{X},q} \circ \overline{c}_{1,f}\right)^{-1}\left(\partial\overline{X}\right)$$

for all $n \geqslant \log_q(r)$.

Let $N(f)$ be the smallest integer greater than $\log_q(r)$. Then there exists $d \in \mathbb{N}$ such that $1 + 1/d < q^{N(f)}/r$. For such a choice of d and all $n \geqslant N(f)$,

$$\overline{c}_{2,f}^{-1}\left(\left(\partial\overline{X}\right)_{d+1}\right) \subseteq \overline{c}_{1,f}^{-1}\left(\left(\partial\overline{X}\right)_{r(d+1)}\right) \subseteq \overline{c}_{1,f}^{-1}\left(\left(\partial\overline{X}\right)_{q^n d}\right) \subseteq \left(F^n_{\overline{X},q} \circ \overline{c}_{1,f}\right)^{-1}\left(\left(\partial\overline{X}\right)_d\right).$$

Thus $\partial\overline{X}$ is $[f]^{(n)}$-contracting for all $n \geqslant N(f)$.

By repeated use of Proposition 3.2.1, for all $k \geqslant 1$ one can find a self correspondence $[\widetilde{f^k}]$ of $\overline{X}$ (with $[\widetilde{f}] := [\overline{f}]$) which is a compactification of $\Gamma^t_{f^k}$ and such that $[\widetilde{f^k}]^{(nk)}$ is contracting along $\partial\overline{X}$ for all $n \geqslant N(f)$ (see Remark 3.2.3).

Let $\mathcal{F} := j_!\mathbb{Q}_\ell$ be the sheaf on $\overline{X}$ supported on X. For any $n \geqslant 1$, Lemma 3.1.11 implies that, $[\widetilde{f^k}]^{(nk)}$ is a compactification of $[\Gamma^t_{F^{nk}_{X,q} \circ f^k}]$. Hence for any $n \geqslant 1$, as in Example 1.2.8, the cohomological correspondence

$$u_k^{(n)} : (F^{nk}_{X,q} \circ f^k)^*\mathbb{Q}_\ell \to \mathbb{Q}_\ell \text{ (lifting } [\Gamma^t_{F^{nk}_{X,q} \circ f^k}]),$$

extends to a cohomological self-correspondence $\overline{u}_k^{(n)}$ of $\mathcal{F}$ (lifting $[\widetilde{f^k}]^{(nk)}$).

Now suppose n is an integer greater than or equal to $N(f)$.

Since $\mathcal{F}$ is supported on X and $[\widetilde{f^k}]^{(nk)}$ is contracting along $\partial\overline{X}$, Corollary 3.1.17 implies that,

$$\sum_{\beta \in \pi_0(\mathrm{Fix}([\widetilde{f^k}]^{(nk)}))} LT_\beta(\overline{u}_k^{(n)}) = \sum_{\beta \in \pi_0(\mathrm{Fix}([\widetilde{f^k}]^{(nk)}|_X))} LT_\beta(\overline{u}_k^{(n)}|_X). \tag{3.3.2}$$

The correspondence $[\widetilde{f^k}]^{(nk)}|_X$ is $\Gamma^t_{F^{nk}_{X,q} \circ f^k}$ and by Lemma 1.3.6 the connected components of $\mathrm{Fix}([\widetilde{f^k}]^{(nk)}|_X)$ are just the fixed points of $F^{nk}_{X,q} \circ f^k$. Moreover [8], Corollary 2.2.4 (b) implies that $[\widetilde{f^k}]^{(nk)}|_X$ is contracting near every closed point in a neighbourhood of fixed points. Hence Theorem 3.1.15 (2) implies that the local terms at the each of the fixed points of $F^{nk}_{X,q} \circ f^k$ is precisely 1. Hence the sum on the right hand side of (3.3.2) is precisely the number of fixed points of $F^{nk}_{X,q} \circ f^k$. The result now follows from the Lefschetz-Verdier trace formula (see Corollary 1.2.11) applied to the cohomological correspondence $\overline{u}_k^{(n)}$ lifting the proper correspondence $[\widetilde{f^k}]^{(nk)}$, and using the observation made in Example 1.2.8.

□

Remark 3.3.2. It follows from Remark 3.2.4 that for a given positive integer k, $kN(f)$ is possibly not the optimal integer for which one has the desired trace formula (see (3.3.1)). However from the point of view of iteration it is natural to derive a trace formula as in Theorem 3.3.1 using this (possibly sub-optimal) bound.

Example 3.3.3. Here we construct an explicit example with $N(f) > 1$. We look at Example 2.4.2 carefully. We continue using the same notations.

As shown there, $\mathrm{Fix}(f \circ F^n_{T,q}) = |P_f(q^n)|$ and $\mathrm{Tr}((f \circ F^n_{T,q})^*, H^*_c(T, \mathbb{Q}_\ell)) = P_f(q^n)$ for all integers $n \geqslant 1$. Hence if f is chosen such that $\det(M_f) = 1$ and $\mathrm{Tr}(M_f) > q + \frac{1}{q}$. Then $N(f)$ *is necessarily greater than* 1.

For $k \geqslant 1$, let M_{f^k} be the linear map on the co-character lattice $X^*(T)$ induced by f^k. Let $P_{f^k}(t) := \det(1 - tM_{f^k}; X^*(T)) \in \mathbb{Z}[t]$.

We now show that $N(f)$ can be chosen to be the smallest positive integer satisfying $P_f(q^{N(f)}) \geqslant 0$. That is $P_f(q^{N(f)}) \geqslant 0$ implies that $P_{f^k}(q^{kN(f)}) \geqslant 0,\ \forall k \geqslant 1$. Note that

$$P_{f^k}(q^{kN(f)}) \geqslant 0 \iff q^{kN(f)} + \tfrac{1}{q^{kN(f)}} \geqslant \mathrm{Tr}(M_{f^k}).$$

Let α and β be the complex eigenvalues of M_f. Then one has the following possibilities,

(1) α (and hence β) is a real number and $\max(|\alpha|, |\beta|) > 1$.

(2) α (and hence β) is a real number and $|\alpha| = |\beta| = 1$.

(3) α is not a real number and hence $\beta = \overline{\alpha}$ and $|\alpha| = 1$.

Suppose we are in the situation of (2) or (3) above, then for all $k \geqslant 1$,

$$q^{kN(f)} + \tfrac{1}{q^{kN(f)}} \geqslant 2 = |\alpha|^k + |\beta|^k \geqslant \mathrm{Tr}(M_{f^k}).$$

Now suppose we are in the situation of case (1). That is, α and β are real, and at least one of them (say α) has modulus greater than 1.

Suppose α is negative then so is β, and for any odd $k \geqslant 1$, one trivially has

$$q^{kN(f)} + \tfrac{1}{q^{kN(f)}} \geqslant \mathrm{Tr}(M_{f^k}).$$

Thus we are reduced to the case when either α is positive and $k \geqslant 1$ or α is negative and k is an even number greater or equal to 2. Hence, after squaring, we can assume that α is positive and greater than 1.

The real valued functions $f_k : (1, \infty) \to \mathbb{R}^+$ given by $x \mapsto x^k + \frac{1}{x^k}$ are *strictly increasing* for all $k \geqslant 1$. Hence $f_k(x) \geqslant f_k(y)$ for some $k \geqslant 1$ and some $x, y \in (1, \infty)$ implies $f_k(x) \geqslant f_k(y)$ for all $k \geqslant 1$. Since by assumption $f_1(q^{N(f)}) \geqslant f_1(\alpha)$, one has $q^{kN(f)} + \frac{1}{q^{kN(f)}} = f_k(q^{N(f)}) \geqslant f_k(\alpha) = \mathrm{Tr}(M_{f^k})$ for all $k \geqslant 1$.

Now we give an alternative proof for the density of periodic points for surjective and proper self-maps of varieties over finite fields (see [9]). The density is true even for non-proper maps and was derived as a Corollary to more general result on the intersection of a correspondence with the graph of the (geometric) Frobenius (see [11], Corollary 0.4 and [10]).

We work over k, an algebraic closure of a finite field $\mathbb{F}_q$.

A variety (over k) is a finite type, separated and integral scheme over k.

We shall need the following standard Lemma which we state here without a proof.

Lemma 3.3.4. *Let $f : X \to Y$ be a proper surjective morphism between varieties of same dimension over k. Then, the induced morphism on the top degree compactly supported cohomology is multiplication by the generic degree of f.*

Definition 3.3.5. Let $f : X \to X$ be a self map of a finite type scheme over k. A closed point $x \in X$ is said to be a *periodic point*, if it is a fixed point of f^k for some $k \geqslant 1$.

Theorem 3.3.6. *Let X_0 be a scheme over $\mathbb{F}_q$ and f_0 a proper, surjective self morphism of X_0 (over $\mathbb{F}_q$). Then, the set of periodic points of f_0 is Zariski dense in X_0.*

Proof. Let (X, f) be the base change of (X_0, f_0) to k.

We can assume X_0 is reduced (and hence geometrically reduced). Also, the statement is true for the pair (X_0, f_0) iff it is true for some $(X_{0,m}, f_{0,m}^n)$ where $(X_{0,m}, f_{0,m})$ is the base change of (X_0, f_0) to the finite sub-field $\mathbb{F}_{q^m}$ of k and n is positive integer.

Since f is dominant, replacing f by a suitable iterate we can ensure that each irreducible component of X is mapped onto itself. Further each of these components are defined over a finite sub-field of k. Hence if necessary, after replacing f_0 by a suitable iterate and a finite extension of the base field, one can assume that each component of X_0 is geometrically integral and is stabilized by f_0.

Thus, we are reduced to the case when f_0 is a dominant and proper self map of a geometrically integral scheme X_0 of dimension d.

Let Z_0 be the Zariski closure of the periodic points of X_0 with the reduced induced structure. Let Z be the closed subscheme of X obtained by base changing Z_0 to k. Then f_0 restricts to a proper and dominant self morphism of Z_0. Since f_0 and $F_{X_0,q}$ commute and any closed point is a fixed point of $F_{X_0,q}$, we deduce that, a closed point is f_0-periodic iff it is $(f_0 \circ F_{X_0,q})$-periodic.

Let $\pi : X \to X_0$ be the map induced by base changing X_0 to k. A closed point is $(f_0 \circ F_{X,q})$-periodic iff any point on the fiber of π over it is a fixed point of $(f \circ F_{X,q})^r$ for some positive integer r. A similar statement is also true for $f_0|_{Z_0}$. Since Z_0 is the Zariski closure of the periodic points of f_0, Theorem 3.3.1 implies that there exists an integer $N >> 0$ such that for all $n \geqslant N$,

$$Z(X_0, f_0 \circ F_{X_0,q}^{n-1}) = Z(Z_0, (f_0)|_{Z_0} \circ F_{Z_0,q}^{n-1}) = \exp(\sum_{r \geqslant 1} \frac{\mathrm{Fix}(f^r \circ F_{X,q}^{nr}) t^r}{r}) \in \mathbb{Q}[[t]], \quad (3.3.3)$$

where $\mathrm{Fix}(f^r \circ F_{X,q}^{nr})$ is the number of fixed points of $f^r \circ F_{X,q}^{nr}$ acting on X.

Let $\lambda \geqslant 1$, be the generic degree of f_0. Fix an embedding,

$$\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}. \quad (3.3.4)$$

Let $\lambda_{\max}$ be the maximum of the spectral radii (with respect to τ) for the actions of f^* and $f|_Z^*$ on the compactly supported ℓ-adic cohomology of X and Z respectively. There exists an integer $N' \geqslant 0$ such that for all integers $n \geqslant N'$, one has

$$\lambda_{\max} q^{n(d-\frac{1}{2})} < q^{nd} \leqslant q^{nd}\lambda. \tag{3.3.5}$$

Consider the Zeta functions $Z(X_0, f_0 \circ F^{n-1}_{X_0,q})$ and $Z(Z_0, (f_0)|_{Z_0} \circ F^{n-1}_{X_0,q})$ for any $n \geqslant \max\{N, N'\}$. Then (3.3.3) implies that these Zeta functions are the same, and hence by Corollary 1.4.4 have the same meromorphic continuation as rational functions (with coefficients in $\mathbb{Q}$) to the entire complex plane. Let $\frac{R(t)}{Q(t)}$ be this analytic continuation, where $R(t)$ and $Q(t)$ are co-prime rational polynomials.

Let

$$P_{i,X}(t) := \det(1 - t(f \circ F^n_{X,q})^*, H^i_c(X, \mathbb{Q}_\ell)) \text{ and}$$

$$P_{i,Z}(t) := \det(1 - t(f|_Z \circ F^n_{Z,q})^*, H^i_c(Z, \mathbb{Q}_\ell)).$$

Thus one has,

$$Z(X_0, f_0 \circ F^{n-1}_{X_0,q}) = \prod_{i=0}^{2d} P_{i,X}(t)^{(-1)^{i+1}} = \frac{R(t)}{Q(t)} \text{ in } \mathbb{C}[[t]] (\text{via } \tau \text{ in } (3.3.4)) \tag{3.3.6}$$

and

$$Z(Z_0, (f_0)|_{Z_0} \circ F^{n-1}_{Z_0,q}) = \prod_{i=0}^{2\dim(Z)} P_{i,Z}(t)^{(-1)^{i+1}} = \frac{R(t)}{Q(t)} \text{ in } \mathbb{C}[[t]] (\text{via } \tau \text{ in } (3.3.4)). \tag{3.3.7}$$

Let α be any complex root of $\prod_{i=0}^{d-1} P_{2i+1,X}(t)$. Since, the i^{th} compactly supported ℓ-adic cohomology is of weight less than or equal to i ([23] Théorème 1 (3.3.1)), (3.3.5) implies that

$$\tfrac{1}{|\alpha|} < q^{nd}\lambda.$$

Further the one-dimensional $\mathbb{Q}_\ell$-vector space $H^{2d}_c(X, \mathbb{Q}_\ell)$ is pure of weight d, hence Lemma 3.3.4 implies that $P_{2d,X}(t) = (1 - q^{nd}\lambda t)$. Thus (3.3.6) implies that $Q(t)$ *has a zero at* $t = \frac{1}{q^{nd}\lambda}$.

Suppose Z_0 is not equal to X_0, then Z is necessarily of smaller dimension than X. Since the i^{th} compactly supported ℓ-adic cohomology is of weight less than or equal to i, any complex root α' of $\prod_{i=0}^{\dim(Z)} P_{2i,Z}(t)$ satisfies

$$\tfrac{1}{|\alpha'|} \leqslant q^{n(d-1)}\lambda_{\max}.$$

It follows then from (3.3.5) that, for any such root α' one has $|\alpha'| > \frac{1}{q^{nd}\lambda}$. In particular, (3.3.7) implies that, $Q(t)$ does not have a zero at $t = \frac{1}{q^{nd}\lambda}$. This is a contradiction. $\square$

Chapter 4

The case of a smooth projective variety

In this chapter, we build on an idea of O'Sullivan as developed by Truong in [30] and obtain a Gromov-Yomdin type bound on the spectral radius for the action of a self-map of a smooth projective variety over an arbitrary base field on its ℓ-adic cohomology.

Throughout this chapter we will work over an algebraically closed field k. Let ℓ be a prime, co-prime to the characteristic of k. We fix, once and for all an isomorphism of $\mathbb{Q}_\ell(1)$ with $\mathbb{Q}_\ell$. Hence we will talk of cycles classes with values in ℓ-adic cohomology without the Tate twist. We also fix an embedding

$$\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}. \tag{4.0.1}$$

A variety (over k) is a finite type, separated and integral scheme over k.

4.1 Some preliminaries from intersection theory

Let X be any smooth, projective variety over k.

Let $Z^*(X)$ be the free abelian group generated by the set of closed subvarieties of X and graded by co-dimension (see [31] Section 1.3). Let $A^*(X)$ be the graded (by co-dimension) Chow ring of X (see loc. cit. Section 8.3). The group underlying $A^*(X)$ is a graded quotient of $Z^*(X)$ by rational equivalence. We shall write $A(X) := \oplus_i A^i(X)$ when we want to ignore the grading and the ring structure.

The *components* of an algebraic cycle $[Z] \in Z^*(X)$ are the subvarieties of X which appear in $[Z]$ with non-zero coefficients. To any closed *subscheme* $Y \subseteq X$ we can associate an effective cycle $[Y]$ in $Z^*(X)$ whose components are precisely the irreducible components of Y (see [31] Section 1.5).

Let $A^*_{\text{num}}(X)$ (respectively $A^*_{\text{num}}(X)_{\mathbb{Q}}$, respectively $A^*_{\text{num}}(X)_{\mathbb{R}}$) be the graded (by codimension) ring of algebraic cycles on X modulo numerical equivalence, with $\mathbb{Z}$ (respectively $\mathbb{Q}$, respectively $\mathbb{R}$) coefficients (see [32] Section 1.1).

Let $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$ be the graded (by codimension) ring of algebraic cycles on X modulo homological equivalence (with respect to ℓ-adic cohomology), with $\mathbb{Q}$ coefficients (see [15] Chapitre 4, [19] Chapter 6 for a construction of cycle classes). Note that $A^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ is a quotient of $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$, which in turn is a $\mathbb{Q}$-subalgebra of $\oplus_i H^{2i}(X, \mathbb{Q}_\ell)$.

For a morphism $f : X \to Y$ of smooth, projective varieties over k, there is a pull-back map $f^* : A^*(Y) \to A^*(X)$ and a push-forward map $f_* : A(X) \to A(Y)$ (see [31] Proposition 8.3 (a) and Theorem 1.4). The pull-back is a morphism of graded rings and the push-forward is a morphism of abelian groups. Further they satisfy a projection formula (loc. cit. Proposition 8.3 (c)). In particular there exists a group homomorphism $\pi_{X*} : A(X) \to A(\mathrm{Spec}\,(k)) \simeq \mathbb{Z}.[\mathrm{Spec}\,(k)]$.(see loc. cit. Definition 1.4).

$A^*_{\mathrm{num}}(X)$ and $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$ also have similar functorial properties (see [32] Section 1).

We shall denote the intersection product on these rings by a '.'. For cycles $[Z]$ and $[Z']$ of complimentary co-dimension in X, by abuse of notation we shall denote the integer $\pi_{X*}([Z].[Z'])$ by $[Z].[Z']$.

Let $[\mathbb{P}^s_k] \in A^{n-s}(\mathbb{P}^n_k),\ 0 \leqslant s \leqslant n$ be the class of a s-dimensional linear sub-space of $\mathbb{P}^n_k$. The Chow ring $A^*(\mathbb{P}^n_k)$ is isomorphic to the graded ring $\mathbb{Z}[x]/(x^{n+1})$ under the map $[\mathbb{P}^{n-1}] \to x$ (see [31] Proposition 8.4) and the class $[\mathbb{P}^s_k]$ generates abelian group $A^{n-s}(\mathbb{P}^n_k),\ 0 \leqslant s \leqslant n$ (see [31] Example 1.9.3).

Definition 4.1.1. The *degree* of $[Z] \in A^s(\mathbb{P}^n_k)$ is the integer $[Z].[\mathbb{P}^s_k]$. For a subvariety $Z \hookrightarrow \mathbb{P}^n_k$ by $\deg(Z)$ we mean $\deg([Z])$.

For any two smooth, projective varieties X and Y (over k), there is an exterior product map (see [31] Section 1.10) $A^*(X) \otimes_{\mathbb{Z}} A^*(Y) \to A^*(X \times_k Y)$, which is a morphism of graded rings (see [31] Example 8.3.7). We shall denote the image of $[Z] \otimes [Z']$ by $[Z] \times [Z']$.

In what follows, we will need a bound (see Proposition 4.1.9) well known to experts and proved using standard techniques. For ease of exposition we present a proof along the lines of [34] (compare [31] Example 11.4.1, [33] Lemma 2.2).

Definition 4.1.2. Two subvarieties V and W in a smooth projective variety X are said to *intersect properly*, if the each component of $V \cap W$ has the right dimension (i.e. $\dim(V) + \dim(W) - \dim(X)$).

Remark 4.1.3. In a similar vein, cycles $[V]$ and $[W]$ in $Z^*(X)$ are said to *intersect properly* if each component of $[V]$ intersects each component of $[W]$ properly.

Suppose now $i : X \hookrightarrow \mathbb{P}^n_k$ is a closed embedding of a smooth, projective variety of dimension r.

Fulton's definition of intersection multiplicities implies the following statement (see [31] Section 6.2, Section 7.1).

Proposition 4.1.4. *Let $[C] \in Z^*(\mathbb{P}^n_k)$ be a cycle on $\mathbb{P}^n_k$ which intersects $[X]$ properly. Then,*

$$i^*([C]) = \sum_j i(Z_j; [X], [C])[Z_j] \in A^*(X),$$

where Z_j's are the irreducible components of the intersection of X with the components of $[C]$, and $i(Z_j; [X], [C])$'s are the intersection multiplicities along the Z_j's (see [31] Definition 7.1).

Remark 4.1.5. By abuse of notation the cycle $\sum_j i(Z_j; X, C)[Z_j] \in Z^*(X)$ will also be denoted by $[C].[X]$. Moreover if $[C]$ is an effective cycle so is $[C].[X]$ (see [31] Proposition 7.1).

Let $V \subseteq X$ be a closed subvariety of dimension d. Let $L \subseteq \mathbb{P}^n_k$ be a linear subspace of dimension $n - r - 1$ disjoint from X.

We denote by $C_L(V) \subseteq \mathbb{P}^n_k$, the cone of V (see [34] Section 2) over L or equivalently the join of V and L (see [31] Example 8.4.5). It is a subvariety of dimension $n+d-r$, and of degree equal to the degree of V (see loc. cit. Example 8.4.5). Moreover V is an irreducible component of $C_L(V) \cap X$ and every component of $C_L(V) \cap X$ is of dimension equal to d (see [34] Lemma 2).

Remark 4.1.6. Hence for any such L, we see that $C_L(V)$ and X intersect properly (see Defintion 4.1.2) and $[C_L(V)].[X]$ denotes the corresponding cycle on X (see Remark 4.1.5).

For an arbitrary cycle $[V] = \sum_i m_i[V_i] \in Z^{r-d}(X)$ we define

$$[C_L([V])] := \sum_i m_i[C_L(V_i)] \in Z^{r-d}(\mathbb{P}^n).$$

Let V and W be closed subvarieties of X. We define the *excess* of V (relative to W) to be 0 if they do not intersect. Else it is defined to be the maximum of the (non-negative) integers

$$\dim(Y) - \dim(V) - \dim(W) + \dim(X),$$

where Y runs through all the components of $V \cap W$. We denote the excess by $e(V)$. For a cycle $[V] := \sum_i m_i[V_i]$ in $Z^*(X)$, we define $e([V]) := \sum_i m_i e(V_i)$.

We have the following result from [34] (used there to prove the "Chow moving Lemma").

Lemma 4.1.7. *(see [34] Main Lemma)*

Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective closed subvariety of dimension r. Let W be a subvariety of X. For any cycle $[V] \in Z^*(X)$, there exists a dense open subset U of $G(n, n-r-1)$, the Grassmanian of linear sub-spaces in $\mathbb{P}^n$ of dimension $n-r-1$, such that for any closed point $x \in U$, if L_x denotes the corresponding linear subspace, then:

(1) $L_x \cap X = \varnothing$.

(2) $e\left([C_L([V])].[X]-[V]\right) \leqslant max(e([V]-1,0))$. *Here the excess is calculated with respect to* W*.*

Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective closed subvariety of dimension r. Let V and W be closed subvarieties of X. Let d be the dimension of V.

The following Lemma is now easy to deduce.

Lemma 4.1.8. *There exists a positive integer* $k \leqslant r+1$ *and a sequence of effective cycles* $\{[V_j]\}_{0\leqslant j\leqslant k}$ *and* $\{[E_j]\}_{1\leqslant j\leqslant k}$ *in* $Z^{r-d}(X)$ *such that,*

(1) $[V_0] = [V]$ *in* $Z^{r-d}(X)$*.*

(2) $[V_j] = [E_{j+1}] - [V_{j+1}]$ *in* $Z^{r-d}(X)$ *for all* $0 \leqslant j \leqslant k-1$*.*

(3) For all $j \geqslant 1$*, the* $[E_j]$*'s are 'ambient' cycles that is,* $[E_j] = i^*\left(deg([V_{j-1}])\,[\mathbb{P}^{n-d+r}_k]\right)$ *in* $A^{r-d}(X)$*.*

(4) Every component of $[V_{k-1}]$ *and* $[V_k]$ *intersects* W *properly (see Definition 4.1.2).*

In particular

$$[V] = \textstyle\sum_{j=1}^{k}(-1)^{j+1}[E_j] + (-1)^k[V_k] \text{ in } Z^{r-d}(X).$$

Proof. Let

$$[V_0] := [V] \in Z^{r-d}(X).$$

For any integer $j \geqslant 1$, having defined $[V_{j-1}] \in Z^{r-d}(X)$ and proven that it is effective, we define

$$[E_j] := [C_{L_j}([V_{j-1}])].[X] \in Z^{r-d}(X) \tag{4.1.1}$$

where L_j is linear sub-space of $\mathbb{P}^n$ of dimension $n-r-1$ (see Remark 4.1.6), chosen such that

$$e(i^*[C_{L_j}([V_{j-1}])] - [V_{j-1}]) \leqslant \max(e([V_{j-1}]-)1, 0) \text{ (see Lemma 4.1.7).}$$

Here the excess is with respect to W. Since $[C_{L_j}(V_{j-1})]$ and $[X]$ intersect properly (see Definition 4.1.2 and [34] Lemma 2), Remark 4.1.5 implies that $[E_j]$ is an effective cycle.

For any integer j having defined $[V_{j-1}]$ and $[E_j]$, we define,

$$[V_j] := [E_j] - [V_{j-1}] \text{ in } Z^{r-d}(X).$$

Since for any subvariety $V \subseteq X$, V is an irreducible component of $C_L(V) \cap X$ (see [34] Lemma 2), the effectivity of $[V_j]$ for any $j \geqslant 1$, is a consequence of the effectivity of $[E_j]$.

Since $e([V_0]) = e([V]) \leqslant r$, for any $j \geqslant r$, the excess $e([V_j]) = 0$. Let $k-1$ be the smallest integer j with the property that $e([V_{k-1}]) = 0$. Then every component of the algebraic cycles $[V_{k-1}]$ and $[V_k]$ intersects W properly.

For any $j \geqslant 1$ since $C_L([V_{j-1}])$ and X intersect properly (see [34] Lemma 2), Proposition 4.1.4 implies that

$$[E_j] = i^* \left([C_L([V_{j-1}])]\right) \in A^{r-d}(X). \tag{4.1.2}$$

For any $j \geqslant 1$ since $[C_L([V_{j-1}])]$ as a cycle on $\mathbb{P}^n_k$ has degree equal to the degree of $[V_{j-1}]$ (see [31] Example 8.4.5), thus (4.1.2) implies that,

$$[E_j] = i^* \left(\deg\left([V_{j-1}]\right)[\mathbb{P}^{n-d+r}_k]\right) \text{ in } A^{r-d}(X).$$

□

Now we derive a basic estimate which is needed later.

Proposition 4.1.9. *Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective variety. There exists a constant C depending only on X such that, for any two closed subvarieties V, W of complimentary dimension in X, $|[V].[W]| \leqslant C\, deg(i_*([V]))\, deg(i_*[W])$.*

Proof. We use Lemma 4.1.8 to construct a sequence of algebraic cycles $\{[V_j]\}_{0 \leqslant j \leqslant k}$ and $\{[E_j]\}_{1 \leqslant j \leqslant k}$ in $Z^{r-d}(X)$ where d is the co-dimension of V in X and satisfying properties (1)-(4) in Lemma 4.1.8.

Since,

$$[V] = \textstyle\sum_{j=1}^{k}(-1)^{j+1}[E_j] + (-1)^k[V_k] \text{ in } Z^{r-d}(X),$$

one has that

$$|[V].[W]| \leqslant \sum_{j=1}^{k} |[E_j].[W]| + |[V_k].[W]|. \tag{4.1.3}$$

Note that $[E_j] = i^* \left(\deg([V_{j-1}])[\mathbb{P}^{n+d-r}_k]\right)$ (see Lemma 4.1.8 (3)) and hence for every $j \geqslant 1$,

$$[E_j].[W] = \deg(W)\deg([V_{j-1}]). \tag{4.1.4}$$

Since every component of $[V_{k-1}]$ intersects $[W]$ properly, $[V_k].[W]$ is bounded above by $[E_k].[W] = \deg(W)\deg([V_{k-1}])$ (see [31] Proposition 7.1). Combining (4.1.3) and (4.1.4) we get,

$$|[V].[W]| \leqslant \left(\sum_{j=1}^{k} \deg([V_{j-1}]) + \deg([V_{k-1}]) \right) \deg(W). \tag{4.1.5}$$

Projection formula implies that for every $j \geqslant 1$,

$$\deg([E_j]) = \deg(X)\deg([V_{j-1}]).$$

Since the $[E_j]$'s and $[V_j]$'s are effective,

$$\deg([V_j]) \leqslant \deg([E_j]) = \deg(X)\deg([V_{j-1}]).$$

Thus for every $j \geqslant 1$

$$\deg([V_j]) \leqslant \deg(X)^j \deg(V) \leqslant \deg(X)^{r+1}\deg(V). \tag{4.1.6}$$

Thus (4.1.5) and (4.1.6) together imply that

$$|[V].[W]| \leqslant (r+2)\deg(X)^{r+1}\deg(V)\deg(W).$$

□

4.2 Gromov algebra

Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a smooth, projective variety over an algebraically closed field k.
Let $[H] \in A^1(X)$ be the class of a hyperplane section.
Let ω be the cohomology class of $[H]$ in $H^2(X, \mathbb{Q}_\ell)$.
For $j \geqslant 1$, let $[H]^j$ denote the j^{th} self-intersection (in $A^*(X)$) of $[H]$.
Let $f : X \to X$ be a self-map of X/k.
For integers $j, m \geqslant 1$ let

$$\delta_j(f^m) := [H]^{r-j}.f^{m*}([H]^j) = f^{m*}([H]^j).[H]^{r-j}. \tag{4.2.1}$$

We have a commutative diagram,

$$\begin{tikzcd}
 & X \times_k X \arrow[d, "1_X \times f^m"] & \\
X \arrow[ur, hook, "\Delta_X"] \arrow[r, hook, "\Gamma_{f^m}"] & X \times_k X \arrow[r, hook, "i \times i"] & \mathbb{P}^n_k \times_k \mathbb{P}^n_k
\end{tikzcd}.$$

Here Γ_{f^m} is the graph of f^m.

Lemma 4.2.1. *Using the above notations,*

$$((i \times i) \circ \Gamma_{f^m})_* ([X]) = \sum_{j=0}^{r} \delta_{r-j}(f^m)([\mathbb{P}^{r-j}_k] \times [\mathbb{P}^j_k]) \tag{4.2.2}$$

Proof. Clearly we can assume $m = 1$. Let $[Y] := \left((i \times i) \circ \Gamma_f\right)_* ([X])$.

The exterior product map $A^*(\mathbb{P}^n_k) \otimes_{\mathbb{Z}} A^*(\mathbb{P}^n_k) \to A^*(\mathbb{P}^n_k \times_k \mathbb{P}^n_k)$ is an isomorphism of graded rings (see [31] Example 8.3), hence

$$[Y] = \textstyle\sum_{j=0}^r n_j([\mathbb{P}^{r-j}_k] \times [\mathbb{P}^j_k]), \text{ where for any } j \geqslant 0,\ n_j = \big([Y].\left([\mathbb{P}^j_k] \times [\mathbb{P}^{r-j}_k]\right)\big).$$

The projection formula and the commutative diagram above imply that

$$n_j = [Y].\left([\mathbb{P}^j_k] \times [\mathbb{P}^{r-j}_k]\right) = [X].\Gamma_f^*\left([H]^j \times [H]^{r-j}\right) = [X].\Delta_X^*\left([H]^j \times f^*\left([H]^{r-j}\right)\right).$$

The definition of intersection product (see [31] Section 8.1 and Corollary 8.1.3) implies that

$$[X].\Delta_X^*\left([H]^j \times f^*\left([H]^{r-j}\right)\right) = \delta_{r-j}(f).$$

Thus for any $m \geqslant 1$, $\left((i \times i) \circ \Gamma_{f^m}\right)_* ([X]) = \sum_{j=0}^r \delta_{r-j}(f^m)([\mathbb{P}^{r-j}_k] \times [\mathbb{P}^j_k])$. □

Definition 4.2.2. The *homological Gromov algebra* $\mathrm{A}^{Gr}_{\mathrm{hom}}(f, \omega)_{\mathbb{Q}}$ is the smallest f^*-stable sub-algebra of $A^*_{\mathrm{hom}}(X)_{\mathbb{Q}}$ containing ω.

Definition 4.2.3. The *numerical Gromov algebra* $\mathrm{A}^{Gr}_{\mathrm{num}}(f, [H])_{\mathbb{Q}}$ is the smallest f^*-stable sub-algebra of $A^*_{\mathrm{num}}(X)_{\mathbb{Q}}$ containing $[H]$.

The numerical Gromov algebra with real coefficients $\mathrm{A}^{Gr}_{\mathrm{num}}(f, [H])_{\mathbb{R}}$ is the $\mathbb{R}$-algebra $\mathrm{A}^{Gr}_{\mathrm{num}}(f, [H])_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$.

Let λ_i be the spectral radius of f^* acting on $A^i_{\mathrm{num}}(X)_{\mathbb{Q}},\ 0 \leqslant i \leqslant \dim(X)$.

Let χ_i be the spectral radius of f^* acting on $A^i_{\mathrm{hom}}(X)_{\mathbb{Q}},\ 0 \leqslant i \leqslant \dim(X)$.

Let μ_j be the spectral radius (with respect to τ in (4.0.1)) of f^* acting on $H^j(X, \mathbb{Q}_\ell),\ 0 \leqslant j \leqslant 2\dim(X)$.

Let λ^{Gr} and χ^{Gr} be the spectral radii of f^* acting on $\mathrm{A}^{Gr}_{\mathrm{num}}(f, [H])_{\mathbb{Q}}$ and $\mathrm{A}^{Gr}_{\mathrm{hom}}(f, \omega)_{\mathbb{Q}}$ respectively.

Note that λ^{Gr} is also the spectral radius of f^* acting on $\mathrm{A}^{Gr}_{\mathrm{num}}(f, [H])_{\mathbb{R}}$.

The following lemma is obvious.

Lemma 4.2.4. *Using the above notations we have inequalities,*

$$\lambda^{Gr} \leqslant \max_{0 \leqslant i \leqslant dim(X)} \lambda_i \leqslant \max_{0 \leqslant i \leqslant dim(X)} \chi_i \leqslant \max_{0 \leqslant j \leqslant 2dim(X)} \mu_j.$$

Further

$$\lambda^{Gr} \leqslant \chi^{Gr} \leqslant \max_{0 \leqslant i \leqslant dim(X)} \chi_i.$$

Lemma 4.2.5. *Let $\{a_{m,i}\}_{m \geqslant 1}, 1 \leqslant i \leqslant s$ be a collection of sequences of complex number. Let $b_i,\ i \leqslant i \leqslant s$ be non-zero complex numbers. Then,*

$$\limsup_m |\sum_{i=1}^s a_{m,i}b_i|^{1/m} \leqslant \max_{1\leqslant i\leqslant s} \limsup_m |a_{m,i}|^{1/m}.$$

Proof. For every i replacing the sequence $\{a_{m,i}\}_{m\geqslant 1}$ by $\{a_{m,i}b_i\}_{m\geqslant 1}$ we can assume without any loss of generality that $b_i = 1$ for all i, since for any $b \neq 0$, $\lim_{m\to\infty} |b|^{1/m} = 1$.

The claimed inequality is obvious for $s = 1$. Suppose that $s \geqslant 2$.

Let $\{a'_{m,2}\}_{m\geqslant 1} := \{\sum_{i=2}^s a_{m,i}\}_{m\geqslant 1}$. Then $|\sum_{i=1}^s a_{m,i}b_i|^{1/m} = |a_{m,1} + a'_{m,2}|^{1/m}$. Hence the claimed inequality is true for a collection of s sequences iff it is true for a collection of $s-1$ sequences. Hence we are reduced to the case when $s = 2$.

Note that

$$\limsup_m |a_{m,1} + a_{m,2}|^{1/m} = \limsup_m \frac{|a_{m,1} + a_{m,2}|^{1/m}}{2^{1/m}} = \limsup_m (\frac{|a_{m,1} + a_{m,2}|}{2})^{1/m}. \quad (4.2.3)$$

Without any loss of generality we can assume that

$$a := \limsup_m |a_{m,1}|^{1/m} \geqslant \limsup_m |a_{m,2}|^{1/m}.$$

Hence for any $\epsilon > 0$ there exists an integer $M >> 0$ such that for all $m \geqslant M$ and $i = 1, 2$,

$$|a_{m,i}| \leqslant (a + \epsilon)^m.$$

In particular

$$|a_{m,1} + a_{m,2}| \leqslant 2(a + \epsilon)^m \text{ for all } m \geqslant M.$$

Thus

$$\limsup_m (\tfrac{|a_{m,1}+a_{m,2}|}{2})^{1/m} \leqslant \limsup_m (a + \epsilon) = a + \epsilon \text{ for any } \epsilon > 0.$$

Then (4.2.3) implies the required bound.

□

Let V be any finite dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$) and $T : V \to V$ a linear map. Let $||.||$ be any matrix norm.

We have the following theorem of Gelfand (see [20] Theorem 18.9)

Theorem 4.2.6. $\limsup_m ||T^m||^{1/m} = \rho(T)$, *where $\rho(T)$ is the spectral radius of T.*

Though the following can be possibly deduced by other standard results, we attempt to give an elementary argument.

Lemma 4.2.7. *Let μ_i, $1 \leqslant i \leqslant n$ be complex numbers of unit modulus. Then for any $\epsilon > 0$, there exist infinitely many integers m such that, for every integer $i \in [1, n]$ one has $|\mu_i^m - 1| < \epsilon$. In particular for any such m, $Re(\mu_i^m) > 1 - \epsilon$.*

Proof. First we observe that, given any $\epsilon > 0$, it suffices to produce *one* m which does the job. This is because, by making ϵ smaller we can then produce infinitely many such m.

Let p be any integer greater than $\frac{2\pi}{\epsilon}$. Then we can cover the unit circle by p many arcs $J_i,\ 1 \leqslant i \leqslant p$, each which has an arc length less than ϵ.

Let T^n be the real torus of dimension n. Then T^n can be covered by p^n sets, each of the form $J_{i(1)} \times J_{i(2)} \cdots J_{i(n)}$ with $1 \leqslant i(1),\ i(2) \cdots, i(n) \leqslant p$.

Consider the infinite sequence of (possibly non distinct) points $\{(\mu_1^k, \mu_2^k,\ \cdots,\ \mu_n^k)\}_{k \geqslant 1} \in T^n$. Clearly there exist distinct positive integers k and k' such that both $(\mu_1^k, \mu_2^k,\ \cdots,\ \mu_n^k)$ and $(\mu_1^k, \mu_2^k,\ \cdots,\ \mu_n^k)$ belong to $J_{i(1)} \times J_{i(2)} \cdots J_{i(n)}$, for some indices $1 \leqslant i(1),\ i(2) \cdots, i(n) \leqslant p$ (this includes the case when the all the μ_i's are roots of unity).

Clearly $m = |k - k'|$ does the job.

□

Let K be a normed field such that, there exists an embedding $\tau : K \hookrightarrow \mathbb{C}$ of normed fields.

Let V be any finite dimensional vector space over K and $T : V \to V$ a linear map. Then,

Proposition 4.2.8. $\limsup_m |\mathit{Tr}(T^m)|^{1/m} = \rho(T)$, *where* $\rho(T)$ *is the spectral radius of* T.

Proof. Using the embedding τ and base changing to $\mathbb{C}$, we can assume that V is a complex vector space, and T is a linear operator on V.

Since we are over $\mathbb{C}$, Theorem 4.2.6 (with the ℓ^1-norm) implies that

$$\limsup_m |\mathrm{Tr}(T^m)|^{1/m} \leqslant \rho(T).$$

Thus it suffices to prove the reverse inequality. Clearly we can assume T has at least one non-zero eigenvalues.

Let $\lambda_i,\ 1 \leqslant i \leqslant n$ be the collection of non-zero eigenvalues of T.

Let $\mu_i := \frac{\lambda_i}{|\lambda_i|}$ be complex numbers with unit modulus.

Lemma 4.2.7 shows that there exist infinitely many m such that, for every integer $i \in [1, n]$ one has $\mathrm{Re}(\mu_i^m) > \frac{1}{2}$.

Then for any such m

$$|\mathrm{Tr}(T^m)| \geqslant \mathrm{Re}(\mathrm{Tr}(T^m)) \geqslant \textstyle\sum_i \mathrm{Re}(\lambda_i^m) \geqslant \frac{\sum_i |\lambda_i|^m}{2} \geqslant \frac{\rho(T)^m}{2}.$$

Thus $\limsup_m |\mathrm{Tr}(T^m)|^{1/m} \geqslant \rho(T)$ and we get the required equality.

□

Now we prove the principal result of this chapter.

Theorem 4.2.9. *Let X be a smooth, projective variety over an arbitrary algebraically closed field k. Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a closed embedding and $[H] \in A^1(X)$ (respectively $\omega \in H^2(X, \mathbb{Q}_\ell)$) be the class of an hyperplane section in the Chow group (respectively ℓ-adic cohomology). Let $f : X \to X$ be a self-map of X/k. Then all the inequalities in Lemma 4.2.4 are in fact equalities.*

Thus, the spectral radius of f^* acting on $H^*(X, \mathbb{Q}_\ell)$ with respect to $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ is independent of τ, and coincides with the spectral radius of f^* on the numerical Gromov algebra.

Proof. Suppose $\dim(X) = r$. Clearly it suffices to show that $\lambda^{\mathrm{Gr}} \geqslant \mu_j,\ 0 \leqslant j \leqslant 2r$.

Recall that $\delta_j(f^m) = [H]^{r-j}.f^{m*}([H]^j) = f^{m*}([H]^j).[H]^{r-j}$ (see Definition (4.2.1)).

We shall first show that for any integer $i \in [0, 2r]$,

$$\mu_i \leqslant \max_{0 \leqslant j \leqslant r} \limsup_m |\delta_j(f^m)|^{1/m}. \tag{4.2.4}$$

It is clear from the definitions of μ_i and $\delta_j(f^m)$ that they specialise well, and thus it suffices to prove the bound (4.2.4), when k is an algebraic closure of a finite field (see for example the proof of Theorem 2.3.5, where such a reduction to the case of a finite field is carried out).

Hence we now assume that k is an algebraic closure of a finite field.

For any integer $m \geqslant 1$, let $[\Gamma_{f^m}] \in A^r_{\mathrm{num}}(X \times X)$ be the cycle corresponding to the graph of f^m.

The work of Katz-Messing ([26] Theorem 2.1) and the Lefschetz trace formula (see [35] Section 3.3.3) implies that, for every integer $i \in [0, 2r]$, there exist an algebraic cycle $\pi^i_X \in Z^r(X \times X)_{\mathbb{Q}}$ (the i^{th} 'Kunneth component') such that,

$$\mathrm{Tr}(f^{m*}; H^i(X, \mathbb{Q}_\ell)) = (-1)^i [\Gamma_{f^m}].\pi^{2r-i}_X, \tag{4.2.5}$$

representing the trace as an intersection product (on the product variety $X \times_k X$).

Recall that we have fixed an embedding $\tau : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ (see (4.0.1)). Thus $\mathbb{Q}_\ell$ is a normed field via this embedding.

Proposition 4.2.8 and (4.2.5) together imply that,

$$\mu_i = \limsup_m |[\Gamma_{f^m}].\pi^{2r-i}_X|^{1/m},\ 0 \leqslant i \leqslant 2r. \tag{4.2.6}$$

There exist finitely many subvarieties $W^{2r-i}_j \subseteq X \times_k X$ of codimension r (the components of the 'Kunneth components') and a constant C' such that for every $m \geqslant 1$,

$$|[\Gamma_{f^m}].\pi^{2r-i}_X| \leqslant C' \sum_j |[\Gamma_{f^m}].[W^{2r-i}_j]|,\ 0 \leqslant i \leqslant 2r. \tag{4.2.7}$$

Note that we have the Segre embedding $X \times_k X \hookrightarrow \mathbb{P}^{n^2+2n}_k$.

The estimate in Proposition 4.1.9 (applied to the smooth projective variety $X \times_k X \subseteq \mathbb{P}_k^{n^2+2n}$) and (4.2.7) imply that there exists a constant C'' (depending only on $i : X \hookrightarrow \mathbb{P}_k^n$ and the choice of Kunneth components), such that for every $m \geqslant 1$,

$$|[\Gamma_{f^m}].\pi_X^{2r-i}| \leqslant C''\deg([\Gamma_{f^m}])(\sum_j \deg(W_j^{2r-1})),\ 0 \leqslant i \leqslant 2r. \tag{4.2.8}$$

The degree in (4.2.8) is with respect to the embedding $X \times_k X \hookrightarrow \mathbb{P}_k^{n^2+2n}$. Moreover Lemma 4.2.2 implies that

$$\deg(\Gamma_{f^m}) = \textstyle\sum_{j=0}^r \delta_{r-j}(f^m)\deg([\mathbb{P}_k^{r-j}] \times [\mathbb{P}_k^j]).$$

Hence (4.2.6) and (4.2.8) together with Lemma 4.2.3 imply that, for any integer $i \in [0, 2r]$,

$$\mu_i \leqslant \max_{0 \leqslant j \leqslant r} \limsup_m |\delta_j(f^m)|^{1/m}. \tag{4.2.9}$$

Thus we have obtained the bound (4.2.4) over an arbitrary algebraically closed field.

For the rest of the proof we work over the algebraically closed field k, we started with. Let $\mathrm{A}_{\mathrm{num}}^{Gr}(f, [H])_{\mathbb{R}}$ be the numerical Gromov algebra with $\mathbb{R}$-coefficients (see Definition 4.2.3).

Let $||.||$ be any norm on the finite dimensional $\mathbb{R}$-vector space $\mathrm{A}_{\mathrm{num}}^{Gr}(f, [H])_{\mathbb{R}}$. Note that f^* is a graded linear transformation of $\mathrm{A}_{\mathrm{num}}^{Gr}(f, [H])_{\mathbb{R}}$. For every integer $m \geqslant 1$, we denote the norm of the linear map f^{m*} acting on $\mathrm{A}_{\mathrm{num}}^{Gr}(f, [H])_{\mathbb{R}}$ by $||f^{m*}||$.

Recall that $\delta_j(f^m) = f^{m*}([H]^j).[H]^{r-j}$. Since the intersection product is bilinear, the map from the j^{th} graded part of $\mathrm{A}_{\mathrm{num}}^{Gr}(f, [H])_{\mathbb{R}}$ to $\mathbb{R}$, obtained by taking intersection product with $[H]^{r-j}$ is linear. Consequently there exists a constant $\widetilde{C'}$ independent of m, such that for any $m \geqslant 1$,

$$|\delta_j(f^m)| \leqslant \widetilde{C'}||f^{m*}(H^j)||,\ 0 \leqslant j \leqslant r. \tag{4.2.10}$$

Since f^* is a linear map, (4.2.10) implies that there exists a constant $\widetilde{C}$ independent of m such that, for any $m \geqslant 1$,

$$|\delta_j(f^m)|^{1/m} \leqslant \widetilde{C}^{1/m}||f^{m*}||^{1/m},\ 0 \leqslant j \leqslant r. \tag{4.2.11}$$

Thus Theorem 4.2.6, (4.2.9) and (4.2.11) together imply

$$\mu_i \leqslant \lambda^{Gr},\ 0 \leqslant i \leqslant 2r.$$

□

Remark 4.2.10. Note that Theorem 4.2.9 generalizes Theorem 2.2.2 (1) to higher dimensions.

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