Supplement 2

1. Prove that $\cos(1/n) \to 1$ by using the definition only and standard trigonometric equalities.

2. (Hard) Let $x \in \mathbf{R}$ be fixed. Prove that the sequence

$$a_n = \sin(n\pi x), \quad n = 1, 2, \ldots$$

has finitely many subsequential limits (accumulation points) if and only if $x \in \mathbf{Q}$; and every number in $[-1, 1]$ is an accumulation point, if x is irrational.

3. Let $0 \leq a_n$, $a_{n+1} \leq a_n$ for $n = 1, 2, \ldots$, and $a_n \to 0$. Prove that the series

$$a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n + \ldots$$

converges (i.e., that the sequence $s_n = a_1 - a_2 + a_3 - a_4 + \ldots + (-1)^{n+1} a_n$ converges). This is the statement of Theorem 3.43 but you have to provide a different proof along the following lines:

(a) Show that s_{2n} is an increasing sequence and s_{2n+1} is a decreasing sequence, and $s_{2n} \leq s_{2n+1}$, $\forall n$.

(b) Show next that s_{2n} is bounded and therefore converges to its least upper bound a.

(c) Show that $s_n \to a$.