Supplement 3

1. Below, $h(x)$ stands for the Heaviside function defined by $h(x) = 0$ for $x < 0$; $h(x) = 1$ for $x \geq 0$. Which of the following functions is continuous on the indicated set? Do not prove your answer using the ε–δ definition, just explain it by paying special attention to the suspicious point(s).

(a) $f = 1/x$ on $\mathbf{R} \setminus \{0\}$,
(b) $h(x)$ on $\mathbf{R}$,
(c) $h(x)$ on $\mathbf{R} \setminus \{0\}$,
(d) $\sin(1/x)$ on $(0, \infty)$.

2. In each example below, answer whether it is possible to define $f(0)$ in such a way so that the extended function f, originally defined on $\mathbf{R} \setminus \{0\}$, becomes continuous on $\mathbf{R}$. Prove your answer (whether it is yes or not) with the definition of a continuous function. Prove continuity/discontinuity at 0 only using that $\sin(x)$ is continuous and 2π periodic, and the definition

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \ldots.$$

(a) $\sin x/x$.
(b) $\sin(1/x)$.
(c) $x \sin(1/x)$.

3. Prove that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$ using the definition.

4. Let $f : \mathbf{R} \to \mathbf{R}$ be a periodic continuous function ($\exists T > 0$ such that $\forall x$, $f(x + T) = f(x)$). Prove that there exist minimal and maximal values of f on $\mathbf{R}$.

5. Show that if f is continuous and bounded on $\mathbf{R}$, and $K \subset \mathbf{R}$ is a closed set, then f may not have maximum (and minimum) value on K by giving a counter-example.

6. Find an explicit example of a continuous function on $\mathbf{R}$ that does not have the property that f maps any open set into an open set, i.e., find a continuous f and an open U such that $f(U)$ is not open. Construct a similar example, if $f : (-1, 1) \to \mathbf{R}$.