MA-504, Spring 2025, Practice Problems for the final exam

Problem 1. If $\sum a_n$ is absolutely convergent, which of the following is absolutely convergent?

$$\sum \frac{a_n}{1+a_n}, \quad \sum \frac{a_n^2}{1+a_n^2}, \quad \sum \sqrt{|a_n a_{n+1}|}, \quad \sum \frac{|a_n a_{n+1}|}{|a_n|+|a_{n+1}|}$$

Problem 2. Study the convergence/divergence of $\sum x^{\ln n}$ for $x > 0$.

Problem 3. Discuss the uniform convergence of the following functions on the indicated intervals. If possible, find the limit function.

(a) $f_n(x) = 1 - nx$, $x \in \mathbf{R}$;
(b) $f_n(x) = 1 - x/n$, $x \geq 0$;
(c) $f_n(x) = \frac{n^2 x}{1+n^3x^2}$, $x \in \mathbf{R}$;
(d) $f_n(x) = e^{-nx}/n$, $x \geq 0$;
(e) $f_n(x) = nxe^{-nx}$, $x \in [0, 1]$;
(f) $f_n(x) = \frac{nx}{1+nx}$, $x \geq 0$;
(h) $f_n(x) = n^2x^2 - 2nx$ if $0 \leq x \leq 2/n$; $f_n(x) = 0$ for $2/n < x \leq 2$.

Problem 4. Let f be thrice differentiable in $[a, b]$. If $f(a) = f(b) = f'(a) = f'(b) = 0$, prove that there exists $\xi \in (a, b)$ such that $f'''(\xi) = 0$.

Problem 5. Discuss the uniform convergence of the following series on the indicated domains:

(a) $\sum ne^{-nx}$, $x \geq 2$;
(b) $\sum n^{-x}$, $x \geq \sqrt{3}$;
(c) $\sum (nx)^{-2}$, $x \neq 0$;
(d) $\sum (1 - x^2)x^n$, $0 \leq x \leq 1$;
(e) $\sum_{n=0}^{\infty} x^n e^{-nx}$, $x \geq 0$.

Problem 6. Find

$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Hint: Find first $\sum_{n=1}^{\infty} n^2 x^n$.

Problem 7. Construct a power series whose exact interval of convergence is $[-3, 5)$.

Problem 8. Expand $f(x) = \arctan x$ as a power series about $x_0 = 0$ and find the interval of convergence. Hint: study f' first.

Problem 9. Let f be a continuous real valued function on $\mathbf{R}$. Assume that

$$f(x+y) = f(x)f(y), \quad \forall x, y.$$

Prove that $f(x) = a^x$ for some $a \geq 0$.

Problem 10. Let f is defined in $(0,1]$ and has derivative uniformly bounded on the same interval. Show that $f(1/n)$ converges.

Problem 11. Show that $U \subset \mathbf{R}^n$ is open if and only if it is a countable union of open balls (that may intersect).

Problem 12. Let $f(x)$ be a polynomial. Prove that $|f|$ attains a minimal value in $\mathbf{R}$, i.e., that there exists $a \in \mathbf{R}$ such that $|f(a)| \leq |f(x)|$ for any x.

Problem 13. Prove the following weaker version of the fundamental theorem in algebra: A polynomial of n-th degree cannot have more than n real distinct zeros.

Problem 14. Let $f : \mathbf{R} \to \mathbf{R}$ be continuous on $\mathbf{R}$ and differentiable at 0. Assume that $f(0) = 0$. Prove that then the function $f(x)/x$, defined for $x \neq 0$, can be defined at $x = 0$ such that the so extended function is continuous on $\mathbf{R}$. In other words, prove that then $f(x) = xg(x)$ with $g(x)$ continuous on $\mathbf{R}$.

Problem 15. Is the function

$$f(x) = \sum_{n=1}^{\infty} \frac{x\sin(nx) + \ln(n+x)}{n^2}$$

continuous?

Problem 16. Prove a comparison test for uniform convergence on $\mathbf{R}$: if $0 \leq f_n \leq g_n$, $\forall n$, and $\sum g_n$ converges uniformly, then so does $\sum f_n$.

Problem 17. Let f be real-valued and have continuous derivative on $[a,b]$. Assume that $f' \neq 0$ on $[a,b]$. Show that then f^{-1} exists and is differentiable with derivative at y equal to $1/f'(x)$, where $x = f^{-1}(y)$.

Problem 18. Is the function $f(x) = x|x|$ differentiable at $x = 0$?

Problem 19. Let r_n denote the sequence of all rational numbers in $(0,1)$ (there are many different ways to arrange them as a sequence, assume that this is one of them). Show that $\liminf r_n^{r_n} = e^{-1/e}$, $\limsup r_n^{r_n} = 1$.

Problem 20. Show that $x^n/(1+x^n)$ does not converge uniformly on $[0,2]$.

Problem 21. Construct a sequence of functions on $[0, 1]$ each of which is discontinuous at any point of $[0, 1]$ and which converges uniformly to a continuous function.

Problem 22. Let f_n be a sequence of continuous functions on $\mathbf{R}$ that converge uniformly to a function f. Prove that

$$\lim_{n\to\infty} f_n(x + 1/n) = f(x)$$

uniformly on any bounded interval.

Problem 23. Let $f : \mathbf{R} \to \mathbf{R}$ be an infinitely differentiable function in $\mathbf{R}$ (that means that the derivatives of all orders exist everywhere). Let $a < b$. Suppose that there exist $M > 0$, $r > 0$, such that

$$\left|\frac{d^n f}{dx^n}(x)\right| \leq M r^{-n} n!, \quad \text{for all } n = 0, 1, \ldots \text{ and for all } x \in (a, b).$$

Prove that then f is analytic in the interval (a, b), i.e. that for any $x_0 \in (a, b)$, that there exists an infinite series $\sum a_n(x - x_0)^n$ with non-zero radius of convergence $\rho > 0$ that converges to f anywhere in $|x - x_0| < \rho$.

Problem 24. We know that $\sin x$ can be approximated uniformly by polynomials in any compact interval. Prove that $\sin x$ cannot be approximated uniformly by polynomials on $\mathbf{R}$.

Problem 25. Convergent or divergent?

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}$$

Problem 26. Find the limit of a_n above. Hint: Interpret a_n as a Riemannian sum of some integrable function.