Math 523/Fall 2024, Assignment 1

1. (10 pts) Show that the Laplace operator Δ is invariant under orthogonal change of coordinates and translations. In other words, let $\Delta_x = \partial^2_{x_1} + \ldots + \partial^2_{x_n}$ as usual and let $y = Ux + a$ be a change of variables, where U is an orthogonal matrix ($UU^T = U^T U = I$) with constant coefficients and a is a constant vector in $\mathbf{R}^n$. Show that after the change of coordinates, Δ_x transforms into $\Delta_y = \partial^2_{y_1} + \ldots + \partial^2_{y_n}$.

2. (10 pts) Let $u(x, y)$ and $v(x, y)$ solve the Cauchy-Riemann system $u_x = v_y$ and $u_y = -v_x$. Show that both u and v solve the Laplace equation $\Delta u = 0$ in $\mathbf{R}^2$.

3. (10 pts)

(a) Find a linear change of coordinate $t \mapsto T$ that reduces the wave equation $u_{tt} = c^2 \Delta u$ into the wave equation $u_{TT} = \Delta u$.

(b) Find a linear change of coordinates $x \mapsto y$ that reduces the wave equation $u_{tt} = c^2 \Delta_x u$ into the wave equation $u_{tt} = \Delta_y u$.

4. (16 pts) For the Maxwell equations

$$\epsilon \partial_t E = \mathrm{curl} H, \quad \mu \partial_t H = -\mathrm{curl} E, \quad \mathrm{div} E = \mathrm{div} H = 0$$

where $\epsilon > 0$, $\mu > 0$ are constants;

(a) Prove that each component E_j of the electric field E and each component H_j of the magnetic field H solves the scalar wave equation $u_{tt} = c^2 \Delta u$ with $c = 1/\sqrt{\varepsilon \mu}$.

(b) Prove that if $\mathrm{div} E = \mathrm{div} H = 0$ for $t = 0$ only and (E, H) solves the rest of the Maxwell system, then $\mathrm{div} E = \mathrm{div} H = 0$ for any t.

5. (16 pts) Prove that the elastic wave equation in $\mathbf{R}^3$

$$\rho u_{tt} = \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u$$

with ρ, λ and μ positive constants, admits the following solutions. Let $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$ and $c_2 = \sqrt{\mu/\rho}$. Let the scalar function ϕ and the vector function v solve wave equations $\phi_{tt} = c_1^2 \Delta \phi$ and $v_{tt} = c_2^2 \Delta v$. Then the vector function $u = \alpha \nabla \phi + \beta \nabla \times v$ solves the elastic wave equation for any two constants α, β. Note that here $\nabla \times v = \mathrm{curl}\, v$ where $\times$ stands for the vector product in $\mathbf{R}^3$.