MATH 523 FALL 2024, ASSIGNMENT 2

Name:

1 (18 pts). Find the general solution of the equation $au_x + bu_y + c = 0$, where $(a, b) \neq (0, 0)$.

Hint: Make a change of variables such that one of the new coordinate axes is a characteristic line (you have some freedom to choose the other one). The general method of characteristics here would work as well, of course but the point here is that in this simple case, one can do it in a more direct way.

2. (18 pts). Given is the equation $2u_x - u_y = 0$. If possible,
(a) find the general solution;
(b) find all solutions that satisfy the condition $u(x, 0) = x^2$;
(c) find all solutions that satisfy the condition $u = x^2$ on the line $x + 2y = 1$;
(d) find all solutions that satisfy the condition $u = 5$ on the line $x + 2y = 1$.

3. (10 pts). (1999) Solve the initial value problem

$$\begin{aligned} 3u_x + u_y &= u^3 \\ u(x, 0) &= \frac{1}{\sqrt{x}}, \quad x > 0. \end{aligned}$$

4. (14 pts). (1998)
(a) Find a solution of

$$\begin{aligned} xu_y - yu_x &= u + 1 \\ u(x, x) &= x^2 \end{aligned}$$

that is valid for all (x, y) in some neighborhood of $(1, 1)$.

(b) Identify **all** points (x_0, x_0) such that there exists a unique solution of the Cauchy problem (a) in some neighborhood of (x_0, x_0). Explain your reasoning.

5. (14 pts). (1997)
Consider the initial value problem

$$\begin{aligned} uu_x + u_y &= u \\ u(x, 0) &= 3x. \end{aligned}$$

(a) Use an existence and uniqueness theorem to prove that this problem has unique solution in a neighborhood of every point on the initial curve $y = 0$.

(b) Solve the problem.

(c) Compute directly (without using the solution formula) u_x, u_y, u_{xx}, u_{xy}, u_{yy} for $y = 0$. Then compute the same derivatives (without u_{yy}) using the solution formula.

6. (14 pts). Consider the equation

$$2u_x - 3u_y + 2u = 2x.$$

(a) Find the solution of that equation which assumes the value $u = x^2$ on the line $y = -x/2$.

(b) Show that there is no solution to that equation with $u = \phi(x)$ on $y = -3x/2$ unless ϕ has the following special form $\phi(x) = x - 1 + ke^{-x}$. For such ϕ, find the solution u.

Notice that the line $y = -3x/2$ in (b) is characteristic.

7. (16 pts). Solve

$$yu_x - xu_y = 0, \quad y > 0$$

with the condition $u = y$ when $x^2 + 2y^2 = 4$. What is the maximal domain (in $y > 0$) where the solution is valid and determined uniquely by its Cauchy data? (Cauchy data = the initial condition above) Draw a sketch.

8. (14 pts). (1986) Solve the initial value problem

$$\begin{aligned} (y+u)u_x + yu_y &= x - y \\ u(x,1) &= 1 + x. \end{aligned}$$

The answer is $u = x - y + 2/y$, but you have to get it.