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A uniqueness result for the inverse back-scattering problem

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Abstract. Let $q_1(t,x)$, $q_2(t,x)$ belong to $C(\mathbb{R}_t; W^{1,\infty}(\mathbb{R}^3_x))$, $q_i(t,x) = 0$ for $|x| > \rho$ with some $\rho > 0$ and let $K_i^\#$ be the corresponding generalized scattering kernels, $i = 1,2$. We prove that if $q_1 \geqslant q_2$ and if $K_1^\#(s', -\omega_0; s, \omega_0) = K_2^\#(s', -\omega_0; s, \omega_0)$ for some $\omega_0 \in S^2$, then $q_1 = q_2$. As a corollary, we get the following result. Let $V_i(x) \in L^\infty(\mathbb{R}^3)$, V_i has compact support and suppose that $-\Delta + V_i$ has no bound states, $i = 1,2$. Let a_i be the scattering amplitude related to V_i, $i = 1,2$. Suppose that $V_1 \geqslant V_2$ and for some ω_0 we have $a_1(k, -\omega_0, \omega_0) = a_2(k, -\omega_0, \omega_0)$ for all k. Then $V_1 = V_2$. Finally we show that $a(k, -\omega, \omega)$ determines uniquely the convex hull of the support of V.

1. Introduction

This paper is devoted to the problem of recovering the potential from the back-scattering data. We study both the wave equation with a time-dependent potential:

$$u_{tt} - \Delta u + q(t,x)u = 0 \qquad t \in \mathbb{R},\ x \in \mathbb{R}^3 \tag{1}$$

and the stationary Schrödinger equation:

$$(-\Delta + V(x) - k^2)v = 0 \qquad x \in \mathbb{R}^3 \tag{2}$$

with a potential $V(x)$ depending only on x. We assume that q and V have compact spatial supports. Clearly, if q does not depend on t, equations (1) and (2) have equivalent scattering theories at least when $q(x)$ has no bound states (see [1]).

The main object in the scattering theory for (1) is the generalized scattering (echo) kernel $K^\#(s', \omega'; s, \omega)$ (see [2, 3] and the next section), while with (2) we associate the scattering amplitude $a(k, \omega', \omega)$. Here $(s, \omega) \in \mathbb{R} \times S^2$ are the parameters of the incoming wave, while $(s', \omega') \in \mathbb{R} \times S^2$ are those of the outgoing wave. It is well known that knowledge of $a(k, \omega', \omega)$ for all k, ω', ω determines $V(x)$ uniquely (see, e.g., [4]). In [3] we have proved that $q(t,x)$ in (1) is uniquely determined by $K^\#(s', \omega'; s, \omega)$. Both inverse problems are overdetermined and in fact the behaviour of a and $K^\#$ near the forward scattering direction $\omega' = \omega$ is sufficient to recover V and q, respectively.

The inverse back-scattering problem is to recover $q(t,x)$ and $V(x)$ if $K^\#$ and a are given for $\omega' = -\omega$. A formal solution to this problem for (2) has been given in [6] for small V. Major progress in the analysis of the inverse back-scattering problem has been made recently by Eskin and Ralston [7]. They have proved that for an open dense set of potentials (not necessarily of compact support) the mapping $V(x) \to a(k, -\omega, \omega)$ is a local homeomorphism. Nevertheless, the (global) uniqueness of the inverse back-scattering problem remains unsolved. Here we give a partial answer to this problem. Namely we

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prove that if q_1 and q_2 (respectively, V_1 and V_2) have the same back-scattering data, then $q_1 = q_2$ ($V_1 = V_2$) under the additional assumption $q_1 \geqslant q_2$ ($V_1 \geqslant V_2$). However, in contrast to [7], we do not require $q_1 - q_2$ ($V_1 - V_2$) to be small. Finally, we show that the back-scattering amplitude $a(k, -\omega, \omega)$ determines uniquely the convex hull of supp V.

The results of this paper have been announced in [5].

2. Assumptions and main results

We impose the following conditions:

Q1. $q \in C(\mathbb{R}_t; W^{1,\infty}(\mathbb{R}^3_x))$, $q = \bar{q}$;

Q2. there exists some $\rho > 0$ such that $q(t,x) = 0$ for $|x| > \rho$; and respectively,

V1. $V \in L^\infty(\mathbb{R}^3)$, $V = \bar{V}$;

V2. V has compact support.

Our principal results are the following theorems.

Theorem 1. Let q_i satisfy conditions Q_1 and Q_2 and denote by $K_i^\#$ the corresponding generalized scattering kernels, $i = 1, 2$. Let $q_1(t,x) \geqslant q_2(t,x)$ for $t_1 \leqslant t \leqslant t_2$, all x, where $t_2 - t_1 > 4\rho$. Suppose there exist $\omega_0 \in S^2$, $d > 2\rho$, such that

$$K_1^\#(s', -\omega_0; s, \omega_0) = K_2^\#(s', -\omega_0; s, \omega_0)$$

for $|s' - s| < d$, $t_1 + \rho < -s < t_2 + \rho$. Then $q_1(t,x) = q_2(t,x)$ for $t_1 + 2\rho \leqslant t \leqslant t_2 - 2\rho$, all x.

As a consequence, if $q_1 \geqslant q_2$ for all t, x and if $K_1^\# = K_2^\#$ for all s', s and $\omega' = -\omega = -\omega_0$, then $q_1 = q_2$ for all t, x.

Now suppose that q in (1) does not depend on t, i.e. we have $q = V(x)$ with some V satisfying conditions V1 and V2. Further suppose that $-\Delta + V$ possesses no bound states. Then the scattering operator S related to (1) (see section 6) exists and the kernel of $S - Id$ in the Lax–Phillips translation representation is given by $S^\#(s' - s, \omega', \omega) = K^\#(s', \omega'; s, \omega)$. Moreover, up to some multiplication factors, the scattering amplitude a related to V coincides with the Fourier transform of $S^\#(s, \omega', \omega)$ with respect to s, namely [3]:

$$-\frac{k}{2\pi \mathrm{i}} a(k, \omega', \omega) = \int \mathrm{e}^{-\mathrm{i}ks} S^\#(s, \omega', \omega)\,\mathrm{d}s.$$

This fact enables us to prove the following.

Theorem 2. Let $V_i(x)$ satisfy V1 and V2 and suppose that $-\Delta + V_i$ has no bound states, $i = 1, 2$. Let a_i be the scattering amplitude related to V_i, $i = 1, 2$. Suppose that $V_1 \geqslant V_2$ and for some ω_0 we have

$$a_1(k, -\omega_0, \omega_0) = a_2(k, -\omega_0, \omega_0) \qquad \text{for all } k.$$

Then $V_1 = V_2$.

Let $\mu(\omega) = \inf\{x \cdot \omega;\ x \in \operatorname{supp} V\}$ be the support function of supp V. The technique developed for the proof of the theorems above enables us to get the following.

Corollary 3. Let $V \geqslant 0$ satisfy V1 and V2. Then

$$-2\mu(\omega) = \sup \operatorname{supp}_s S^{\#}(s, -\omega, \omega). \tag{3}$$

Therefore, $S^{\#}(s, -\omega, \omega)$ (respectively $a(k, -\omega, \omega)$) determines uniquely the convex hull of $\operatorname{supp} V$.

It should be noted that (3) is well known in the case of scattering by a compact obstacle [8]. Moreover, in the obstacle case $S^{\#}(s, -\omega, \omega)$ has a singularity at $s = -2\mu(\omega)$. In the potential case under consideration $S^{\#}(s, -\omega, \omega)$ is smooth if V is smooth, so that there are no singularities. Nevertheless, for non-negative V we have $S^{\#}(s, -\omega, \omega) = 0$ for $s > -2\mu(\omega)$, while

$$S^{\#}(-2\mu - \varepsilon, -\omega, \omega) = -\frac{1}{8\pi^2}\partial_\varepsilon^2\left(\int_{\mu \leqslant x\cdot\omega \leqslant \mu+\varepsilon/2} V(x)\mathrm{d}x(1 + \mathrm{O}(\varepsilon))\right) \qquad \text{as } \varepsilon \to +0$$

where $\mu = \mu(\omega)$. The proof of corollary 3 is based on the above relation.

3. Preliminary

Denote by $\mathscr{H}$ the closure of $C_0^\infty(\mathbb{R}^3) \times C_0^\infty(\mathbb{R}^3) \ni \boldsymbol{f} = (f_1, f_2)$ with respect to the energy norm

$$\|\boldsymbol{f}\|^2 = \tfrac{1}{2}\int_{\mathbb{R}^3} (|\nabla f_1|^2 + |f_2|^2)\mathrm{d}x.$$

$\mathscr{H}$ is a Hilbert space with energy scalar product [9]. Let $U_0(t)$ be the unitary group related to the Cauchy problem for the free wave equation $\Box u = 0$, where $\Box = \partial_t^2 - \Delta_x$. Fix q satisfying Q1 and Q2. There exists a two-parameter family $U(t, s)$ of bounded operators in $\mathscr{H}$, such that the solution of (1) with Cauchy data $(u, u_t) = \boldsymbol{f}$ for $t = s$ is given by $\boldsymbol{u}(t) = U(t, s)\boldsymbol{f}$ [2, 3, 10, 11]. Here and in what follows we denote by $\boldsymbol{u}(t)$ the pair $(u(t, \cdot), u_t(t, \cdot))$ for given function $u(t, x)$. The family $U(t, s)$ is jointly continuous in t, s; $U(t, s)U(s, r) = U(t, r)$ for all t, s, r; and

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t, s)\boldsymbol{f} = (A - Q(t))U(t, s)\boldsymbol{f} \qquad \frac{\mathrm{d}}{\mathrm{d}s} U(t, s)\boldsymbol{f} = -U(t, s)(A - Q(s))\boldsymbol{f}$$

for $\boldsymbol{f} \in D(A)$, where A is the generator of $U_0(t)$, given by $A\boldsymbol{f} = (f_2, \Delta f_1)$, $D(A) = \{\boldsymbol{f} \in \mathscr{H};\ (f_2, \Delta f_1) \in \mathscr{H}\}$ and $Q(t)\boldsymbol{f} = (0, q(t, \cdot)f_1)$ (see [3, 10, 11]). The principle of causality, which is valid for $U_0(t)$ and $U(t, s)$, implies that $\operatorname{supp} U(t, s)\boldsymbol{f} \cup \operatorname{supp} U_0(t - s)\boldsymbol{f} \subset \{x;$ there exists $y \in \operatorname{supp} \boldsymbol{f}$ such that $|x - y| \leqslant |t - s|\}$. This allows us to extend $U_0(t)$ and $U(t, s)$ on the space $\mathscr{H}_{\mathrm{loc}} = \{\boldsymbol{f} \in \mathscr{H};\ \varphi\boldsymbol{f} \in \mathscr{H}$ for any $\varphi \in C_0^\infty\}$.

Below we are going to recall briefly the definition of the generalized scattering kernel $K^{\#}(s', \omega'; s, \omega)$. Set $h_n(\xi) = \xi^n/n!$ for $\xi \geqslant 0$, $h_n(\xi) = 0$ otherwise, $n = 0, 1, \ldots$. Then $h_n' = h_{n-1}$, and h_0 is the Heaviside function. Given $(s, \omega) \in \mathbb{R} \times S^2$, let $w^{\pm}(t, x; s, \omega)$ be the solutions of the Cauchy problems

$$\begin{cases} (\Box + q(t, x))w^+ = 0 \\ w^+|_{t < -s-\rho} = h_1(t + s - x\cdot\omega) \end{cases} \qquad \begin{cases} (\Box + q(t, x))w^- = 0 \\ w^-|_{t > -s+\rho} = h_1(-t - s + x\cdot w) \end{cases}$$

i.e. $\boldsymbol{w}^{\pm}(t; s, \omega) = U(t, -s \mp \rho)(h_1(-\rho \mp x\cdot\omega), \pm h_0(-\rho \mp x\cdot\omega))$. Note that the principle of causality yields $w^+(t, x; s, \omega) = 0$ for $t + s < x\cdot\omega$, while $w^-(t, x; s, \omega) = 0$ for

$t + s > x \cdot \omega$. The function $w_{sc}^{+} = w^{+} - h_1(t + s - x \cdot \omega)$ has an asymptotic wave profile $w_{sc}^{+,\#}(s', \omega'; s, \omega)$, i.e. the limit

$$w_{sc}^{+,\#}(s', \omega'; s, \omega) = \lim_{t \to \infty} t\partial_t w_{sc}^{+}(t, (t + s')\omega'; s, \omega)$$

exists in $L^2_{loc}(\mathbb{R}_{s'} \times S^2_{\omega'})$. We refer to [12] for the definition and the basic properties of the asymptotic wave profiles. It should be noted that if $\boldsymbol{u}(t) = U_0(t)\boldsymbol{f}$ is a free solution, then the asymptotic wave profile

$$u^{\#}(s, \omega) = \lim_{t \to \infty} t\partial_t u(t, (t + s)\omega)$$

exists in $L^2(\mathbb{R}_s \times S^2_\omega)$ and we have $u^{\#} = -\mathscr{R}\boldsymbol{f}$, $\mathscr{R}\boldsymbol{f}$ being the translation representer of $\boldsymbol{f}$ [9].

The generalized scattering kernel $K^{\#}$ is given by

$$K^{\#}(s', \omega'; s, \omega) = -\frac{1}{2\pi}\partial_s^2 w_{sc}^{+,\#}(s', \omega'; s, \omega)$$

(see [2, 3]). The kernel $K^{\#}$ may also be characterized as follows. Let $u^{+}(t, x; s, \omega) = \partial_s^2 w^{+}(t, x; s, \omega)$ be the solution of the problem

$$(\Box + q(t, x))u^{+} = 0$$
$$u^{+}|_{t < -s-\rho} = \delta(t + s - x \cdot \omega).$$

Then in an appropriate sense we have the following asymptotic expansion:

$$\partial_t u^{+}(t, x; s, \omega) = \delta'(t + s - x \cdot \omega) - \frac{2\pi}{|x|} K^{\#}\left(|x| - t, \frac{x}{|x|}; s, \omega\right) + o\left(\frac{1}{|x|}\right)$$

as $t, |x| \to \infty$. If the scattering operator S exists, then $K^{\#}$ is just the Schwartz kernel of $\mathscr{R}(S - Id)\mathscr{R}^{-1}$ [3].

4. Representation of $K_1^{\#} - K_2^{\#}$.

In [3] we derived the following representation of $K^{\#}(s', \omega'; s, \omega)$:

$$K^{\#}(s', \omega'; s, \omega) = -\frac{1}{8\pi^2}\partial_{s'} \iint q(t, x)u^{+}(t, x; s, \omega)\delta(t + s' - x \cdot \omega')\mathrm{d}t\,\mathrm{d}x. \tag{4}$$

The integral above is to be considered in the distribution sense. It is easy to deduce from (4) that $K^{\#}$ is a distribution in s', s depending continuously on ω', ω. Moreover, $K^{\#}(s', \omega'; s, \omega) = 0$ for $s' > s + 2\rho$. Below we are going to derive a similar representation for $K_1^{\#} - K_2^{\#}$, where $K_i^{\#}$ is related to q_i, $i = 1, 2$. Let also $w_i^{\pm}$, u_i, $U_i(t, s)$ be related to q_i, $i = 1, 2$. Denote for simplicity $\boldsymbol{h}(t) = (h_1(t + s - x \cdot \omega),\ h_0(t + s - x \cdot \omega)) \in \mathscr{H}_{loc}$ (which depends also on s, ω). Set

$$p(t, s, \omega) = (\boldsymbol{w}_1^{+}(t; s, \omega) - \boldsymbol{w}_2^{+}(t; s, \omega),\ U_0(t)\boldsymbol{f})_{\mathscr{H}}$$

where $\boldsymbol{f} \in C_0^\infty \times C_0^\infty$ is fixed and $(\ ,\ )_{\mathscr{H}}$ is the scalar product in $\mathscr{H}$. Taking the limit $t \to \infty$, we get immediately

$$\lim_{t \to \infty} p(t, s, \omega) = -\int_{S^2}\int_{\mathbb{R}} [w_{1,sc}^{+,\#}(s', \omega'; s, \omega) - w_{2,sc}^{+,\#}(s', \omega'; s, \omega)]\overline{\mathscr{R}\boldsymbol{f}}(s', \omega')\mathrm{d}s'\,\mathrm{d}\omega'. \tag{5}$$

On the other hand, by Duhamel's expression we get

$$\begin{aligned} w_1^+(t;s,\omega) - w_2^+(t;s,\omega) &= [U_1(t,-s-\rho) - U_2(t,-s-\rho)]\boldsymbol{h}(-s-\rho) \\ &= -\int_{-s-\rho}^{t} U_2(t,\sigma)[Q_1(\sigma) - Q_2(\sigma)]U_1(\sigma,-s-\rho)\boldsymbol{h}(-s-\rho)\mathrm{d}\sigma. \end{aligned}$$

Therefore,

$$p(t,s,\omega) = -\int_{-s-\rho}^{t} (U_2(t,\sigma)[Q_1(\sigma) - Q_2(\sigma)]\boldsymbol{w}_1^+(\sigma;s,\omega), U_0(t)\boldsymbol{f})_{\mathscr{H}}\,\mathrm{d}\sigma.$$

Let $R > 0$ be such that $\operatorname{supp}\boldsymbol{f} \subset B_R = \{x; |x| \leqslant R\}$ and fix $t_0 > R + \rho$. Then $\boldsymbol{f}_+ := U_0(t_0)\boldsymbol{f} \in D_+^{\rho}$, where D_+^{ρ} is the outgoing space introduced by Lax and Phillips [9]. Assume in what follows that $t > t_0$. Then $U_0(t)\boldsymbol{f} = U_0(t-t_0)\boldsymbol{f}_+ = U_2^*(t_0,t)\boldsymbol{f}_+$ (see [10, 12]). Therefore

$$\begin{aligned} p(t,s,\omega) &= \int_{-s-\rho}^{t} (U_2(t_0,\sigma)[Q_1(\sigma) - Q_2(\sigma)]\boldsymbol{w}_1^+(\sigma;s,\omega),\boldsymbol{f}_+)_{\mathscr{H}}\,\mathrm{d}\sigma \\ &= \frac{1}{2}\int_{-s-\rho}^{t_0}\int (q_1(\sigma,x) - q_2(\sigma,x))w_1^+(\sigma,x;s,\omega)\overline{[U_2^*(t_0,\sigma)\boldsymbol{f}_+]_2}\,\mathrm{d}x\,\mathrm{d}\sigma. \end{aligned} \tag{6}$$

Lemma 4. Let $g \in \mathscr{H}$ and denote $v = [U^*(t,s)g]_2$, where $U(t,s)$ is the propagator related to a (possibly complex-valued) potential q. Then v satisfies the following problem in the sense of distributions:

$$\begin{aligned} &(\partial_s^2 - \Delta_x + \overline{q(s,x)})v = 0 \\ &v|_{s=t} = g_2 \qquad v_s|_{s=t} = \Delta g_1. \end{aligned} \tag{7}$$

Proof. Let $\varphi \in C_0^\infty$, $\phi = (0,\varphi) \in \mathscr{H}$. Then

$$\partial_s(v,\varphi)_{L^2} = 2\partial_s(g, U(t,s)\phi)_{\mathscr{H}} = 2(g, U(t,s)(-A + Q(s))\phi\}_{\mathscr{H}}.$$

Since $Q(s)\phi = 0$, we get

$$\partial_s(v,\varphi)_{L^2} = -2(U^*(t,s)g, A\phi)_{\mathscr{H}}.$$

For $s = t$ we have $v|_{s=t} = g_2$ and

$$\partial_{s|_{s=t}}(v,\varphi)_{L^2} = -2(g, A\phi)_{\mathscr{H}} = (g_1, \Delta\varphi)_{L^2}.$$

Therefore, the initial conditions in (7) are satisfied. Furthermore

$$\begin{aligned} \partial_s^2(v,\varphi)_{L^2} &= 2(g, U(t,s)(A^2 - Q(s)A)\phi)_{\mathscr{H}} \\ &= 2(g, U(t,s)(\Delta - q(s,\cdot)\phi))_{\mathscr{H}} \\ &= (v, (\Delta - q(s,\cdot))\varphi)_{L^2}, \end{aligned}$$

hence the first equation in (7) is satisfied, too. The proof is complete. □

Set $v = \overline{[U_2^*(t_0,\sigma)\boldsymbol{f}_+]_2}$. By lemma 4, $(v, v_\sigma) = A\bar{\boldsymbol{f}}_+$ for $s = t_0$ and $(v,v_\sigma) = U_2(\sigma,t_0)A\bar{\boldsymbol{f}}_+ = U_2(\sigma,0)U_2(0,t_0)U_0(t_0)A\bar{\boldsymbol{f}}$. Applying lemma 1 from [2], we get

$$v(\sigma,x) = \frac{1}{2\pi}\int_{S^2}\int_{\mathbb{R}} w_2^-(\sigma,x;s,\omega)\partial_s^2\mathscr{R}(A\bar{\boldsymbol{f}})(s,\omega)\,\mathrm{d}s\,\mathrm{d}\omega.$$

Recall that $\mathscr{R}A\bar{f} = -\partial_s\overline{\mathscr{R}f}$. Substituting the above expression into (6), we get for $t > t_0$

$$p(t,s,\omega) = \frac{1}{4\pi}\int_{S^2}\int_{\mathbb{R}}\int_{\mathbb{R}^3}\int_{\mathbb{R}}(q_1 - q_2)(t,x)w_1^+(t,x;s,\omega)w_2^-(t,x;s',\omega')$$
$$\times\ \partial_{s'}^3(\mathscr{R}\bar{f})(s',\omega')\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}s'\,\mathrm{d}\omega'.$$

Comparing this with (5), we get

$$w_{1,\mathrm{sc}}^{+,\#}(s',\omega';s,\omega) - w_{2,\mathrm{sc}}^{+,\#}(s',\omega';s,\omega)$$
$$= \frac{1}{4\pi}\partial_{s'}^3\iint(q_1 - q_2)(t,x)w_1^+(t,x;s,\omega)w_2^-(t,x;s',\omega')\mathrm{d}t\,\mathrm{d}x$$

therefore

$$K_1^\#(s',\omega';s,\omega) - K_2^\#(s',\omega';s,\omega)$$
$$= -\frac{1}{8\pi^2}\partial_s^2\partial_{s'}^3\iint(q_1 - q_2)(t,x)w_1^+(t,x;s,\omega)w_2^-(t,x;s',\omega')\mathrm{d}t\,\mathrm{d}x. \qquad (8)$$

In particular, if $u_1^+ = \partial_s^2 w_1^+$ and $u_2^- = \partial_s^2 w_2^-$ are the solutions to the problems

$$\begin{cases}(\Box + q_1(t,x))u_1^+ = 0\\ u_1^+|_{t<-s-\rho} = \delta(t + s - x\cdot\omega)\end{cases} \qquad \begin{cases}(\Box + q_2(t,x))u_2^- = 0\\ u_2^-|_{t>-s+\rho} = \delta(-t - s + x\cdot\omega)\end{cases}$$

then we get

$$K_1^\# - K_2^\# = -\frac{1}{8\pi^2}\partial_{s'}\iint(q_1 - q_1)(t,x)u_1^+(t,x;s,\omega)u_2^-(t,x;s',\omega')\mathrm{d}t\,\mathrm{d}x. \qquad (9)$$

Note that (9) reduces to (4) when $q_2 = 0$. Formula (8) (or (9)) is the desired representation of $K_1^\# - K_2^\#$.

5. Proof of theorem 1.

Let q satisfy the regularity assumptions of theorem 1. We need to establish the following result before proceeding.

Lemma 5. If t, x, s and ω run over compact sets, then there exists a constant $C > 0$ such that

$$|w^\pm(t,x;s,\omega) - h_1(\pm(t + s - x\cdot\omega))| \leqslant Ch_2(\pm(t + s - x\cdot\omega)).$$

Proof. Since the proof of lemma 5 follows closely the proof of theorem 3.4 in [11], we shall only sketch it. Iterating Duhamel's expression, we get

$$\boldsymbol{w}^+(t;s,\omega) = U(t,-s-\rho)\boldsymbol{h}(-s-\rho) = \boldsymbol{h}(t) + \sum_{k=1}^{\infty} V_k(t,-s-\rho)\boldsymbol{h}(-s-\rho) \qquad (10)$$

where $\boldsymbol{h}(t) = (h_1(t + s - x\cdot\omega), h_0(t + s - x\cdot\omega))$ and

$$V_k(t,s)\boldsymbol{f} = (-1)^k\int_s^t \mathrm{d}\sigma_1\int_s^{\sigma_1}\mathrm{d}\sigma_2\ldots\int_s^{\sigma_{k-1}}\mathrm{d}\sigma_k\,U_0(t-\sigma_1)Q(\sigma_1)U_0(\sigma_1-\sigma_2)\ldots$$
$$\times\ U_0(\sigma_{k-1}-\sigma_k)Q(\sigma_k)U_0(\sigma_k - s)\boldsymbol{f}.$$

Next, using Kirchoff's formula for $U_0(t)$, we estimate each term in (10) and summing up the corresponding estimates we prove the desired inequality for w^+. The proof for w^- is similar. □

Remark. It should be noted that for $q \in C^\infty$ we have the following singular expansion modulo C^∞:

$$w^+(t,x;s,\omega) \sim h_1(t+s-x\cdot\omega) + \sum_{j=2}^{\infty} A_j(t,x,\omega)h_j(t+s-x\cdot\omega)$$

(see [3]), where $A_j \in C^\infty$ (a similar expression holds for w^-). In this case the lemma follows immediately. Since in these arguments we need not use Kirchoff's formula, the lemma and the results of the paper are valid for arbitrary odd space dimension $n \geqslant 3$, provided $q \in C^\infty$.

Let q_1, q_2 satisfy the assumptions of theorem 1, and put $q = q_1 - q_2$. We have $K_1^\#(s', -\omega_0; s, \omega_0) = K_2^\#(s', -\omega_0; s, \omega_0)$ provided

$$t_1 + \rho < -s < t_2 + \rho \qquad |s'-s| < d. \tag{11}$$

Thus for such s, s' we have $\partial_s^2 \partial_{s'}^3 I(s', -\omega_0; s, \omega_0) = 0$, where I is the integral in (8). Since $I(s', \omega'; s, \omega) = 0$ for $s' > s + 2\rho$ and since $d > 2\rho$, we deduce $I(s', -\omega_0; s, \omega_0) = 0$, i.e.

$$\iint q(t,x) w_1^+(t,x;s,\omega_0) w_2^-(t,x;s',-\omega_0)\,\mathrm{d}t\,\mathrm{d}x = 0 \tag{12}$$

provided that (11) is fulfilled. Set

$$A_\mu = \{(t,x);\ x\cdot\omega_0 + t_1 + \rho \leqslant t \leqslant -x\cdot\omega_0 + t_2 - \rho,\ |x| \leqslant \rho,\ x\cdot\omega_0 \leqslant \mu\}.$$

A_μ is a non-empty compact set (because $t_2 - t_1 > 4\rho$) for $-\rho < \mu < \rho$. Let $\mu_0 = \sup\{\mu; q(t,x) = 0 \text{ for } (t,x) \in A_\mu\}$. Assume that q does not vanish identically on A_μ. Then $-\rho \leqslant \mu_0 < \rho$. Choose $\varepsilon \in (0, \rho - \mu_0)$ and set $s' = s - 2\mu_0 - \varepsilon$. Note that the choice of ε, μ_0 yields $|s' - s| < 2\rho$, thus (11) holds. Therefore, (12) implies

$$\int_{\Omega(s)} q(t,x) w_1^+(t,x;s,\omega_0) w_2^-(t,x;s-2\mu_0-\varepsilon,-\omega_0)\,\mathrm{d}t\,\mathrm{d}x = 0 \tag{13}$$

for $t_1 + \rho < -s < t_2 + \rho$. Assume that $t_1 + \rho < -s < t_2 - \rho - 2\mu_0 - \varepsilon$. Then the integration above is taken over the set

$$\Omega(s) = \{(t,x);\ \mu_0 < x\cdot\omega_0 < \min(t+s, -t-s+2\mu_0+\varepsilon),\ |x| < \rho\}$$

because of the inclusions

$$\operatorname{supp} w_1^+(t,x;s,\omega_0) \subset \{(t,x);\ t+s \geqslant x\cdot\omega_0\}$$

$$\operatorname{supp} w_2^-(t,x;s-2\mu_0-\varepsilon,-\omega_0\} \subset \{(t,x);\ t+s-2\mu_0-\varepsilon \leqslant -x\cdot\omega_0\}$$

$$\operatorname{supp} w_1^+(t,x;s,\omega_0) \cap \operatorname{supp} w_2^-(t,x;s-2\mu_0-\varepsilon,-\omega_0) \subset A_\rho$$

$$\operatorname{supp} q(t,x) \cap A_\rho \subset \{(t,x) x\cdot\omega_0 \geqslant \mu_0,\ |x| \leqslant \rho\}.$$

Let $t_1 + \rho < -s < t_2 - \rho - 2\mu_0 - \varepsilon$, $(t,x) \in \Omega(s)$. Then t, x, s run over bounded sets and according to lemma 5 there exists $\delta > 0$, such that

$$w_1^+(t,x;s,\omega_0) > 0 \qquad \text{for } 0 < t+s-x\cdot\omega_0 \leqslant \delta$$

$$w_2^-(t,x;s-2\mu_0-\varepsilon,-\omega_0) > 0 \qquad \text{for } -\delta \leqslant t+s-2\mu_0-\varepsilon+x\cdot\omega_0 < 0.$$

We may assume that $\varepsilon < \delta$, thus $w_1^+(t, x; s, \omega_0) > 0$ and $w_2^-(t, x; s - 2\mu_0 - \varepsilon, -\omega_0) > 0$ for $(t, x) \in \Omega(s)$, $t_1 + \rho < -s < t_2 - \rho - 2\mu_0 - \varepsilon$. Moreover, we have $q(t, x) \geqslant 0$ for $(t, x) \in \Omega(s)$. Therefore, each integrand in (13) is non-negative and, moreover, w_1^+, w_2^- are positive in $\Omega(s)$. Hence, $q(t, x) = 0$ for $(t, x) \in \Omega(s)$. Letting s run over the interval $t_1 + \rho < -s < t_2 - \rho - 2\mu_0 - \varepsilon$, we get

$$\overline{\bigcup_s \Omega(s)} \cup A_{\mu_0} = A_{\mu_0 + \varepsilon}$$

hence $q = 0$ on $A_{\mu_0 + \varepsilon}$, which contradicts the choice of μ_0. The proof of theorem 1 is complete.

6. The stationary case

Suppose that q in (1) is time independent, i.e. we have $q = V(x)$ with some V satisfying V1 and V2. Then $K^{\#}$ depends merely on $s' - s$ and we have

$$K^{\#}(s', \omega'; s, \omega) = S^{\#}(s' - s, \omega', \omega)$$

where $S^{\#}(s, \omega', \omega) = K^{\#}(s, \omega'; 0, \omega)$. Assume that $-\Delta + V$ has no bound states. Let $U(t)$ be the propagator related to (1) with $q = V$. Then the scattering operator

$$S = s\text{-}\lim_{t \to \infty} U_0(-t) U(2t) U_0(-t)$$

exists [1] and $S^{\#}(s' - s, \omega', \omega)$ is the kernel of $\mathscr{R}(S - Id)\mathscr{R}^{-1}$ [3]. Thus $S^{\#}(s, \omega', \omega)$ is a tempered distribution with respect to s, depending continuously on ω', ω. We have [3]

$$-\frac{k}{2\pi \mathrm{i}} a(k, \omega', \omega) = \int \mathrm{e}^{-\mathrm{i}ks} S^{\#}(s, \omega', \omega) \mathrm{d}s.$$

Therefore, $a(k, -\omega_0, \omega_0)$ determines $K^{\#}(s', -\omega_0; s, \omega_0)$ uniquely. Hence theorem 2 follows from theorem 1 for $V \in W^{1,\infty}(\mathbb{R}^3)$ with compact support. Consider the more general case $q = V \in L^{\infty}(\mathbb{R}^3)$. Then, if $U(t)$ is the group related to (1), we have $(\mathrm{d}/\mathrm{d}t) U(t) f = (A - Q) U(t) f = U(t)(A - Q) f$ for $f \in D(A)$ even under that weaker assumption on q. Thus the proof of theorem 1 goes without any modifications for stationary $q \in L^{\infty}(\mathbb{R}^3)$ with compact support. This completes the proof of theorem 2. □

Finally, let us prove corollary 3. Although it follows immediately from the arguments above, we wish to get more information about the behaviour of $S^{\#}(s, -\omega, \omega)$ near $s = -2\mu(\omega)$. We have the following.

Proposition 6. Let $q \equiv V$ satisfy V1 and V2. Let μ be such that $V(x) = 0$ for $x \cdot \omega \leqslant \mu$. Then $S^{\#}(s, -\omega, \omega) = 0$ for $s > -2\mu$ and if V is non-negative near the plane $x \cdot \omega = \mu$ we have

$$S^{\#}(-2\mu - \varepsilon, -\omega, \omega) = -\frac{1}{8\pi^2} \partial_\varepsilon^2 \left(\int_{\mu \leqslant x \cdot \omega \leqslant \mu + \varepsilon/2} V(x) \mathrm{d}x (1 + \mathrm{O}(\varepsilon)) \right) \qquad \text{as } \varepsilon \to +0.$$

Proof. Since $q = V$ is stationary, we have $w(t, x; s, \omega) = w(t + s, x; 0, \omega)$ and (8) reduces to the following:

$$\begin{aligned} S^{\#}(s, -\omega, \omega) &= -\frac{1}{8\pi^2} \partial_s^5 \iint V(x) w^+(t, x; 0, \omega) h_1(-t - s - x \cdot \omega) \mathrm{d}t \, \mathrm{d}x \\ &= -\frac{1}{8\pi^2} \partial_s^2 \int V(x) \Gamma(-s - x \cdot \omega, x, \omega) \mathrm{d}x. \end{aligned}$$

Here $\Gamma(t, x, \omega) = \partial_t \omega^+(t, x; 0, \omega)$ is the solution of the problem

$$(\square + V)\Gamma = 0 \qquad \Gamma = h_0(t - x \cdot \omega) \text{ for } t < -\rho.$$

Similarly to lemma 5, we have $\Gamma = h_0(t - x \cdot \omega) + \Gamma_{sc}(t, x, \omega)$, where $|\Gamma_{sc}(t, x, \omega)| \leqslant Ch_1(t - x \cdot \omega)$ with $C > 0$ independent of t, x, ω, if t and x run over compact sets. Therefore, if s is bounded we have $S^{\#}(s, -\omega, \omega) = -(I_1 + I_2)/8\pi^2$, where

$$I_1 = \int_{x \cdot \omega < -s/2} V(x)\mathrm{d}x \qquad I_2 = \int_{x \cdot \omega < -s/2} V(x)\Gamma_{sc}(-s - x \cdot \omega, x, \omega)\mathrm{d}x.$$

Hence $I_1 = I_2 = 0$ for $s > -2\mu$ while for $s < -2\mu$ we have

$$|I_2| \leqslant C \int_{\mu < x \cdot \omega < -s/2} V(x) h_1(-s - x \cdot \omega)\mathrm{d}x \leqslant C(-s - 2\mu) \int_{\mu < x \cdot \omega < -s/2} V(x)\mathrm{d}x.$$

Therefore $|I_1 + I_2| \leqslant (1 + \mathrm{O}(\varepsilon))\int_{x \cdot \omega < -s/2} V(x)\mathrm{d}x$. The proof is complete. □

Let us demonstrate corollary 3. Take $\mu = \mu(\omega)$ and assume that $S^{\#}(s, -\omega, \omega) = 0$ for $s < -2\mu - \delta$ with some $\delta > 0$. Then

$$\int_{\mu < x \cdot \omega < \mu + \varepsilon/2} V(x)\mathrm{d}x(1 + \mathrm{O}(\varepsilon)) = 0 \qquad \text{for } 0 < \varepsilon < \delta.$$

Choose δ small enough to have $1 + \mathrm{O}(\varepsilon) > 0$. Then

$$\int_{\mu < x \cdot \omega < \mu + \varepsilon/2} V(x)\mathrm{d}x = 0$$

for all sufficiently small positive ε. Since $V \geqslant 0$ in the integral above, we get a contradiction with the choice of μ.

Remark. Let $q_i = V_i$, $i = 1, 2$ be two stationary potentials satisfying V1 and V2. Then we have (9). We shall show that in this case (9) leads to a useful formula for the difference $a_1 - a_2$ (see (15) below). Since q_i are stationary, it follows that

$$u_1^+(t, x; s, \omega) = \tilde{u}_1^+(t + s, x, \omega) \qquad u_2^-(t, x; s, \omega) = \tilde{u}_2^-(t + s, x, \omega)$$

where $\tilde{u}_1^+$, $\tilde{u}_2^-$ solve the problems

$$\begin{cases} (\square + q_1(x))\tilde{u}_1^+ = 0 \\ \tilde{u}_1^+|_{t < -\rho} = \delta(t - x \cdot \omega) \end{cases} \qquad \begin{cases} (\square + q_2(x))\tilde{u}_2^- = 0 \\ \tilde{u}_2^-|_{t > \rho} = \delta(t - x \cdot \omega). \end{cases}$$

Moreover, we have $\tilde{u}_2^-(t, x, \omega) = \tilde{u}_2^+(-t, x, -\omega)$. Therefore, (9) reduces to the following:

$$S_1^{\#}(s, \omega', \omega) - S_2^{\#}(s, \omega', \omega)$$
$$= -\frac{1}{8\pi^2}\partial_s \iint (V_1 - V_2)(x)\tilde{u}_1^+(t, x, \omega)\tilde{u}_2^+(-t - s, x, -\omega')\mathrm{d}t\,\mathrm{d}x. \qquad (14)$$

Let $\psi_j(k, x, \omega)$ be the generalized eigenfunction of $-\Delta + V_j$ defined as the outgoing solution of the problem

$$(-\Delta - k^2 + V_j(x))\psi_j = 0$$
$$\psi_j = \exp(\mathrm{i}k\omega \cdot x) + \mathrm{O}(|x|^{-1}) \qquad \text{as } |x| \to \infty$$

where $j = 1, 2$. Suppose that $-\Delta + V_j$ have no bound states. Then [3] $\psi_j(k, x, \omega) = \int \exp(ikt)\tilde{u}_j^+(t, x, \omega)dt$. Taking the Fourier transform with respect to s in (14), we obtain

$$a_1(k, \omega', \omega) - a_2(k, \omega', \omega) = -\frac{1}{4\pi}\int (V_1(x) - V_2(x))\psi_1(k, x, \omega)\psi_2(k, x, -\omega')dx. \qquad (15)$$

Note that if $V_2 \equiv 0$, then (15) reduces to the well known formula

$$a_1(k, \omega', \omega) = -\frac{1}{4\pi}\int V_1(x)\psi_1(k, x, \omega)\exp(-ik\omega' \cdot x)\,dx. \qquad (16)$$

Hence, (15) can be considered as a generalization of (16). We hope that (15) might be useful for solving other inverse scattering problems.

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