

# THE LIGHT RAY TRANSFORM FOR PSEUDO-EUCLIDEAN METRICS

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ABSTRACT. We study the ray transform  $L$  over null (light) rays in the pseudo-Euclidean space with signature  $(n', n'')$ ,  $n' \geq 2$ ,  $n'' \geq 2$ . We analyze the normal operator  $L'L$ , derive an inversion formula, and prove stability estimates. We show that the symbol  $p(\xi)$  is elliptic but singular at the light cone. We analyze  $L$  as an Fourier Integral Operator as well. Finally, we compare this to the Minkowski case.

## 1. INTRODUCTION

The purpose of this article is to study the light ray transform in the pseudo-Euclidean space. We consider  $\mathbb{R}^n$ , where  $n = n' + n''$ , with  $n', n'' \geq 2$ . Throughout this article, we trivially identify  $\mathbb{R}^n$  with  $\mathbb{R}^{n'} \times \mathbb{R}^{n''}$ , and denote the points in  $\mathbb{R}^n$  by  $x = (x', x'') \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$ , similarly for vectors and covectors. We assume that the metric  $g$  has signature  $(n', n'')$ , i.e.,

$$g = \text{diag}(-1, \dots, -1, 1, \dots, 1).$$

The case  $n' = 1$  (or  $n'' = 1$  but not  $n' = n'' = 1$  which is not interesting) gives us the Minkowski metric, and the light ray transform for it has been studied already even for more general Lorentzian metrics, see, e.g., [2, 11, 15, 16, 18–21]. Consider the operator  $L$  which integrates  $f$  along null-lines

$$Lf(\gamma) = \int_{\gamma} f \, ds,$$

where  $\gamma$  is a null-line (also called light line/ray) with respect to  $g$ , i.e.,  $g(\dot{\gamma}, \dot{\gamma}) = 0$ , and, say,  $f \in C_0^\infty(\mathbb{R}^n)$ . Recall that any line in  $\mathbb{R}^n$  can be parameterized by

$$t \mapsto x + t\theta, \quad \text{for } x, \theta \in \mathbb{R}^n.$$

For such a line to be a null line, we require  $g(\theta, \theta) = 0$  or  $|\theta'| = |\theta''|$ . Thus, we have the following parameterization for the light ray transform:

$$(1.1) \quad Lf(x, \theta) = \int_{\mathbb{R}} f(x + t\theta) \, dt \quad \text{for } x, \theta \in \mathbb{R}^n, \text{ with } |\theta'| = |\theta''|,$$

and we require below  $x \cdot \theta = 0$  and  $|\theta'| = |\theta''| = 1$ . Thus, the transform studied here is the restriction of the usual ray transform to null-lines with respect to  $g$ , and a generalization of the Light ray transform. It is formally overdetermined by  $(2n - 3) - n = n - 3$  variables.

The transform  $L$  appears naturally as the bicharacteristic one, restricted to functions  $f(x)$  on the base, related to the generalized ultra-hyperbolic operator  $P = \Delta_{x'} - \Delta_{x''}$ , see [14]. In particular, its inversion would recover a compactly supported potential  $V$  from Cauchy data for  $P + V$  on a boundary of a smooth domain containing  $\text{supp } V$ . Also,  $L$  belongs to the class of the generalized Radon transforms, formulated first by Guillemin [6, 7]. Restricted versions of them to  $n$ -dimensional

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complexes of curves has been studied microlocally in [3–5]. Its analytic version has been studied recently in [12] establishing support theorems. The same transform has been studied in [9], where it is shown that  $L$  is injective on  $\mathcal{S}(\mathbb{R}^n)$  using the Fourier Slice Theorem.

In sections 3, 4, we analyze the normal operator  $L'L$  and show that it is a convolution, hence a Fourier multiplier with some  $p(\xi)$  which is positively homogeneous of order  $-1$ , singular at the light cone in  $T^*\mathbb{R}^n$  but still elliptic when either  $n' \geq 3$  or  $n'' \geq 3$ . When  $n' = n'' = 2$ ,  $p(\xi)$  has a logarithmic blow-up at the light cone. In both cases, we present an inversion formula and prove a stability estimate which requires a modified Sobolev space when  $n' = n'' = 2$ . The symbol  $p$  does not seem to have a closed-form formula, except in some particular dimensions (see subsection 3.4), and we present several representations.

In section 5, we analyze  $L$  briefly as a Fourier Integral Operator (FIO) to fit it into the generalized Radon transform framework of Guillemin. We do not get deeply into this analysis since the singularity of  $p(\xi)$  at the light cone suggests the need to use a specialized calculus which is beyond the scope of this work. We refer to the remarks following Theorem 5.1.

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## 2. PRELIMINARIES

**2.1. Setup.** We call the cone  $|\theta'| = |\theta''|$  the light cone; same for the cone  $|\xi'| = |\xi''|$  in the cotangent bundle. If we start with  $\theta$  not restricted to be unit, and non-restricted  $x$  in (1.1), we would have, for  $a \in \mathbb{R} \setminus \{0\}$ ,

$$Lf(x, a\theta) = \frac{1}{|a|} Lf(x, \theta).$$

Due to this scaling property and the fact that  $|\theta'| = |\theta''|$ , we can impose the restriction  $|\theta'| = |\theta''| = 1$ . Thus, we now have

$$\theta = (\theta', \theta'') \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}.$$

Note that we trivially have, analogous to the corresponding property of the X-ray transform, for  $x, \theta \in \mathbb{R}^n$  and  $s \in \mathbb{R}$

$$Lf(x + s\theta, \theta) = Lf(x, \theta).$$

This suggests taking  $x \perp \theta$ . Our operator now is defined as

$$(2.1) \quad Lf(x, \theta) = \int_{\mathbb{R}} f(x + t\theta) dt,$$

where  $\theta = (\theta', \theta'') \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$ ,  $x = (x', x'') \in \mathbb{R}^{n'} \times \mathbb{R}^{n''}$  with  $x \cdot \theta = 0$ . We call that variety  $\Sigma$ , i.e.,

$$(2.2) \quad \Sigma := \left\{ (z, \theta) \mid z \in \mathbb{R}^n, \theta \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}, z \cdot \theta = 0 \right\}.$$

One can think of  $\Sigma$  as the normal bundle of  $\mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$ . We equip it with the measure  $dS(\theta') dS(\theta'') dH_\theta(z)$ , where  $dH_\theta(z)$  is the Euclidean measure on  $\theta^\perp$ .

**2.2. Fourier Slice Theorem.** We now proceed to derive the Fourier Slice theorem for the operator (2.1). To fix our notation, we recall the Fourier transform. For  $f \in \mathcal{S}(\mathbb{R}^n)$ , the Fourier transform is defined as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) dx,$$

and the inverse Fourier transform is given as

$$\mathcal{F}^{-1}f(x) = \check{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.$$

We similarly define the Fourier and inverse Fourier transforms on a hyperplane through the origin, say  $\theta^\perp$ , denoted  $\mathcal{F}_{\theta^\perp}$  and  $\mathcal{F}_{\theta^\perp}^{-1}$  respectively. More precisely, for  $x, \xi \in \theta^\perp$ ,

$$\begin{aligned} \mathcal{F}_{\theta^\perp} f(\xi) &= \int_{\theta^\perp} e^{-ix \cdot \xi} f(x) dH(x), \\ \mathcal{F}_{\theta^\perp}^{-1} f(x) &= (2\pi)^{1-n} \int_{\theta^\perp} e^{ix \cdot \xi} f(\xi) dH(\xi), \end{aligned}$$

where  $dH$  denotes the natural  $n - 1$  dimensional (Lebesgue) measure on  $\theta^\perp$ .

**Theorem 2.1** (Fourier Slice Theorem). *For every  $f \in L^1(\mathbb{R}^n)$ ,*

$$\hat{f}(\xi) = \int_{\theta^\perp} e^{-ix \cdot \xi} Lf(x, \theta) dH(x), \quad \text{whenever } \xi \perp \theta, \theta \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}.$$

The theorem is just a restricted version of the classical Fourier Slice Theorem.

**2.3. Consequences of the Fourier Slice Theorem.** We immediately obtain the following implications of the Fourier Slice Theorem, see also [9].

**Corollary 2.2** (uniqueness).  *$f \in L^1(\mathbb{R}^n)$  and  $Lf = 0$  imply  $f = 0$ .*

*Proof.* We follow [9] here. Notice first that the conormal bundle  $N_\theta^*(\mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1})$  at any  $\theta$  coincides with  $\mathbb{R}^n$ . Indeed, given  $\xi \in \mathbb{R}^n$ , we want to show that the equation  $\xi' \cdot \theta' + \xi'' \cdot \theta'' = 0$  has a solution  $\theta \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$ . Assume, without loss of generality, that  $|\xi'| \leq |\xi''|$ . Choose any  $\theta' \in \mathbb{S}^{n'-1}$ . Then  $|\xi' \cdot \theta'| \leq |\xi'| \leq |\xi''|$ , hence we can find  $\theta'' \in \mathbb{S}^{n''-1}$  so that  $\xi'' \cdot \theta'' = -\xi' \cdot \theta'$ .

By Theorem 2.1,  $\hat{f}(\xi) = 0$  for every  $\xi$ , which completes the proof.  $\square$

### 3. THE NORMAL OPERATOR

#### 3.1. The adjoint and the normal operators.

**Theorem 3.1.** *The transpose of  $L$  is given by*

$$(3.1) \quad L' \phi(z) = \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \phi(z - 2^{-1/2}(z \cdot \theta)\theta, \theta) dS(\theta'') dS(\theta').$$

*The normal operator is*

$$(3.2) \quad L'Lf(x) = 2 \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_0^\infty f(x + t\theta) dt dS(\theta'') dS(\theta').$$

*Proof.* For every  $\phi \in C_0^\infty(\Sigma)$ , we have

$$\begin{aligned} \langle Lf, \phi \rangle &= \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\theta^\perp} Lf(x, \theta) \phi(x, \theta) dH_\theta(x) dS(\theta'') dS(\theta') \\ &= \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\theta^\perp} \int_{\mathbb{R}} f(x + t\theta) \phi(x, \theta) dt dH_\theta(x) dS(\theta'') dS(\theta') \\ &= \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\mathbb{R}^n} f(z) \phi(z - 2^{-1/2}(z \cdot \theta)\theta, \theta) dz dS(\theta'') dS(\theta'). \end{aligned}$$

This proves (3.1). To prove (3.2), write

$$\begin{aligned} (3.3) \quad L'Lf(x) &= \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\mathbb{R}} f\left(x - 2^{-1/2}(x \cdot \theta)\theta + t\theta\right) dt dS(\theta'') dS(\theta') \\ &= \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\mathbb{R}} f(x + t\theta) dt dS(\theta'') dS(\theta'). \end{aligned}$$

Let us split the integral in  $t$  variable into  $\{t < 0\}$  and  $\{t > 0\}$ . Since  $n', n'' \geq 2$ , we can make the changes of variables  $\theta' \mapsto -\theta'$  and  $\theta'' \mapsto -\theta''$  to obtain (3.2).  $\square$

*Remark 3.2.* The above makes sense when (only) one of  $n'$  or  $n''$  equals 1 as well with the integral over  $\mathbb{S}^0 = \{-1, 1\}$  interpreted as a sum, see [18].

We can extend  $L$  to  $\mathcal{E}'(\mathbb{R}^n)$  by duality, as usual.

**3.2. The Schwartz kernel of the normal operator.** Let  $\phi \in C_c^\infty(\mathbb{R}_+)$ . Let us define  $\phi_\epsilon(t) = \epsilon^{-1}\phi(t/\epsilon)$ ,  $\epsilon > 0$ . Then  $\phi_\epsilon \rightarrow \delta$  in  $\mathcal{S}'(\mathbb{R})$ , as  $\epsilon \rightarrow 0$ . For  $f \in \mathcal{S}(\mathbb{R}^n)$ ,

$$L'Lf(x) = 2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_0^\infty \int_{\mathbb{R}} \phi_\epsilon(s) f(x + ((t+s)\theta', t\theta'')) ds dt dS(\theta'') dS(\theta').$$

In the  $(t, \theta'')$  integral, make the change of variables  $y'' = t\theta''$  to get

$$\begin{aligned} L'Lf(x) &= 2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^{n''}} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{R}} \phi_\epsilon(s) \frac{f(x' + (|y''| + s)\theta', x'' + y'')}{|y''|^{n''-1}} ds dS(\theta') dy'' \\ &= 2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^{n''}} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{R}} \phi_\epsilon(s - |y''|) \frac{f(x' + s\theta', x'' + y'')}{|y''|^{n''-1}} ds dS(\theta') dy'' \\ &= 2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^{n''}} \int_{\mathbb{R}^{n'}} \phi_\epsilon(|y'| - |y''|) \frac{f(x' + y', x'' + y'')}{|y'|^{n'-1} |y''|^{n''-1}} dy' dy'', \end{aligned}$$

where in the last equality, we switched to polar coordinates in the first variable. Next,

$$\begin{aligned} (3.4) \quad L'Lf(x) &= 2 \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^{n''}} \int_{\mathbb{R}^{n'}} \phi_\epsilon(|y' - x'| - |y'' - x''|) \frac{f(y)}{|y' - x'|^{n'-1} |y'' - x''|^{n''-1}} dy' dy'' \\ &= 2 \lim_{\epsilon \rightarrow 0} f * \frac{\phi_\epsilon(|\cdot'| - |\cdot''|)}{|\cdot'|^{n'-1} |\cdot''|^{n''-1}}. \end{aligned}$$

We want to interpret the limit above as a delta function. Set  $F(x) = |x'| - |x''|$ . Then  $\nabla F = (x'/|x'|, -x''/|x''|)$ . Therefore,  $F$  is smooth, with  $|\nabla F|^2 = 2$ , in some neighborhood of  $\{F = 0, x \neq 0\}$  not containing the origin. Then away from  $x = 0$ , we have  $\lim_{\epsilon \rightarrow 0} \phi_\epsilon(|x'| - |x''|) = \delta(|x'| - |x''|)$ , which, in particular, is positively homogeneous w.r.t.  $x$  of order  $-1$ . Then the convolution kernel in (3.4) is positively homogeneous of order  $-n+1$  (away from  $x = 0$ ). It has a unique extension to  $\mathbb{R}^n$  as such a distribution, see [8, Theorem 3.2.3]. On the other hand, (3.2) implies that the Schwartz kernel of  $L'L$  is positively homogeneous of order  $-n+1$ , therefore it equals that extension. Indeed, given a function  $h$ , denote temporarily  $h_\lambda = h(\lambda x)$ ,  $\lambda > 0$ . Then by (3.2),  $L'Lf_\lambda = \lambda^{-1}(L'Lf)_\lambda$ ,

$\forall \lambda > 0$ . This implies that the convolution kernel  $K$  of  $L'L$  satisfies  $K_\lambda = \lambda^{1-n}K$ , as claimed. With that understanding, we replace the limit above by a delta (with the understanding that we made sense of the whole expression below, not just of its numerator) to get the following.

**Theorem 3.3.**  *$L'L$  is a convolution with*

$$2 \frac{\delta(|x'| - |x''|)}{|x'|^{n'-1} |x''|^{n''-1}}.$$

The denominator above can be replaced by  $|x'|^{n-2}$  or by  $|x''|^{n-2}$ .

**3.3. The normal operator as a Fourier multiplier.** Since  $L'L$  is a convolution, it is a Fourier multiplier with  $p(\xi)$  equal to the Fourier transform of its convolution kernel. In particular,  $p$  is positively homogeneous of degree  $-1$ . We will compute  $p$  in the following way. By (3.3),

$$\begin{aligned} \widehat{L'Lf}(\xi) &= \int_{\mathbb{R}^n} e^{-ix \cdot \xi} (L'Lf)(x) dx \\ &= \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\mathbb{R}} f(x + t\theta) dt dS(\theta'') dS(\theta') dx. \end{aligned}$$

The multiple integral is not absolutely convergent in general. In order to make it such, we introduce the factor  $e^{-\epsilon t^2/2}$  and eventually we will take the limit  $\epsilon \rightarrow 0+$ . Then the limit of the modified integral would equal the integral above, both considered as distributions in the  $\xi$  variable (as we will see below, we always get a locally  $L^1$  function of  $\xi$  but when  $n' = n'' = 2$ , it is not bounded). With the change of variable  $x \mapsto x - t\theta$ , we obtain

$$\begin{aligned} \widehat{L'Lf}(\xi) &= \lim_{\epsilon \rightarrow 0+} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{-ix \cdot \xi + it\theta \cdot \xi - \epsilon t^2/2} f(x) dx dt dS(\theta'') dS(\theta') \\ &= \widehat{f}(\xi) \lim_{\epsilon \rightarrow 0+} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int_{\mathbb{R}} e^{it\theta \cdot \xi - \epsilon t^2/2} dt dS(\theta'') dS(\theta') \\ &= \widehat{f}(\xi) \lim_{\epsilon \rightarrow 0+} \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} (2\pi/\epsilon)^{1/2} e^{-(\xi \cdot \theta)^2/(2\epsilon)} dS(\theta'') dS(\theta') \\ &= \widehat{f}(\xi) p(\xi), \end{aligned}$$

with the last identity defining  $p(\xi)$  as a distribution. The limit is a delta at  $\xi \cdot \theta$  multiplied by  $2\pi$ , thus the multiplier that we are looking for is given by

$$(3.5) \quad p(\xi) = 2\pi \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \delta(\xi' \cdot \theta' + \xi'' \cdot \theta'') dS(\theta') dS(\theta'').$$

Since  $\psi := \xi \cdot \theta$  has a differential

$$d_\theta \psi = (\xi' - (\xi' \cdot \theta')\theta', \xi'' - (\xi'' \cdot \theta'')\theta'')$$

on  $\mathbb{S}_{\theta'}^{n'-1} \times \mathbb{S}_{\theta''}^{n''-1}$  which vanishes on the null set when  $\xi' = \lambda\theta'$ ,  $\xi'' = -\lambda\theta''$  with some  $\lambda$  (and then  $|\xi'| = |\xi''|$ ). Then the delta above is not correctly defined as a function of  $\theta$  for  $\xi$  on the light cone but the meaning of (3.5), as already mentioned, is

$$(3.6) \quad \langle p, \phi \rangle = 2\pi \int_{\mathbb{S}^{n'-1}} \int_{\mathbb{S}^{n''-1}} \int \delta(\xi \cdot \theta) \phi(\xi) d\xi dS(\theta') dS(\theta''), \quad \forall \phi \in C_0^\infty(\mathbb{R}^n \setminus 0).$$

The inner integral above is in distribution sense, defining a smooth function of  $\theta \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$ . We get in particular that  $p(\xi)$  is  $L^1_{\text{loc}}$  away from the origin since (3.6) defines a measure, and it is easy to see that this measure is absolutely continuous with respect to the Lebesgue one. On the other hand,  $p$  is smooth away from the light cone.

Formally applying the Funk-Hecke theorem

$$\int_{\mathbb{S}^{n-1}} \phi(x \cdot \theta) dS(\theta) = |\mathbb{S}^{n-2}| \cdot |x|^{2-n} \int_{\mathbb{R}} (|x|^2 - t^2)_+^{(n-3)/2} \phi(t) dt, \quad n \geq 2,$$

we find

$$\begin{aligned} p(\xi) &= C_{n',n''} |\xi'|^{2-n'} |\xi''|^{2-n''} \int_{\mathbb{R}} \int_{\mathbb{R}} (|\xi'|^2 - u^2)_+^{(n'-3)/2} (|\xi''|^2 - v^2)_+^{(n''-3)/2} \delta(u+v) du dv \\ (3.7) \quad &= C_{n',n''} |\xi'|^{2-n'} |\xi''|^{2-n''} \int_{\mathbb{R}} (|\xi'|^2 - u^2)_+^{(n'-3)/2} (|\xi''|^2 - u^2)_+^{(n''-3)/2} du, \end{aligned}$$

where  $C_{n',n''} = 2\pi |\mathbb{S}^{n'-2}| \cdot |\mathbb{S}^{n''-2}|$ . Equation (3.7) actually holds in distribution sense as it can be seen by applying the Funk-Hecke theorem to the action of  $p$  on test functions. It defines a function positively homogeneous of order  $-1$  in  $\xi$  as we established earlier, and (3.7) holds in the way written for  $\xi' \neq 0, \xi'' \neq 0$ . The cases of one of them only being zero can be handled as a limit.

**3.4. Examples.** We consider several special cases of  $n'$  and  $n''$ . We use the notation  $p = p_{n',n''}$  below. One of our goals is to get an idea of the singularity across the light cone.

**Example 3.4.** Consider  $n' = n'' = 3$  first. Then

$$p_{3,3}(\xi) = 16\pi^3 \frac{\min(|\xi'|, |\xi''|)}{|\xi'| \cdot |\xi''|} = \frac{16\pi^3}{\max(|\xi'|, |\xi''|)}.$$

Since  $|\xi|/\sqrt{2} \leq \max(|\xi'|, |\xi''|) \leq |\xi|$ , we have

$$\frac{16\pi^3}{|\xi|} \leq p_{3,3}(\xi) \leq \frac{16\sqrt{2}\pi^3}{|\xi|}.$$

In particular, this shows that the singularity at  $\xi = 0$  is locally integrable, thus contributing to a smoothing operator. Next,  $p$  is continuous but not smooth, even cutting away a neighborhood of  $\xi = 0$ ; therefore not a symbol of a pseudo-differential operator. It is such away from the boundary of the characteristic cone  $|\xi'| = |\xi''|$ , described also as  $g^{-1}(\xi, \xi) = 0$ , where  $g^{-1}$  is the co-metric formally looking the same as  $g$ . Its behavior as  $|\xi| \rightarrow \infty$  (but not of its  $\xi$ -derivatives) is like  $\sim |\xi|^{-1}$ . In particular, one gets

$$(3.8) \quad \|f\|_{L^2(\Omega)}/C \leq \|L'Lf\|_{H^1(\mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega),$$

see (4.5). Here, we view  $L^2(\Omega)$  as a subspace of  $L^2(\mathbb{R}^n)$  by using an extension as zero. The proof is the same as in [18].

Recall that

$$p_{n',n''}(\xi) = C_{n',n''} |\xi'|^{2-n'} |\xi''|^{2-n''} \int_{\mathbb{R}} (|\xi'|^2 - u^2)_+^{(n'-3)/2} (|\xi''|^2 - u^2)_+^{(n''-3)/2} du.$$

Set  $\kappa = |\xi'|/|\xi''|$ . Then  $\kappa - 1$  is a defining function of the light cone.

**Example 3.5.** We will investigate  $p_{n',n''}$  when one of the dimensions is 3, say  $n'' = 3$ . Let us assume first that  $\kappa \in (0, 1)$ . As before, we obtain

$$\begin{aligned} p_{n',3}(\xi) &= 2C_{n',3} |\xi''|^{-1} \int_0^1 (1-s^2)^{(n'-3)/2} ds \\ (3.9) \quad &= 2C_{n',3} |\xi''|^{-1} \int_0^{\pi/2} \cos^{n'-2} \theta d\theta = \frac{\sqrt{\pi}\Gamma((n'-1)/2)}{\Gamma(n'/2)} C_{n',3} |\xi''|^{-1}. \end{aligned}$$

Now, let us consider  $\kappa \in (1, \infty)$ . In this case, we find

$$(3.10) \quad \begin{aligned} p_{n',3}(\xi) &= 2C_{n',3}|\xi''|^{-1} \int_0^1 (1 - s^2/\kappa^2)^{(n'-3)/2} ds \\ &= 2C_{n',3}\kappa|\xi'|^{-1} \int_0^{\arcsin(1/\kappa)} \cos^{n'-2} \theta d\theta, \quad \kappa > 1. \end{aligned}$$

Comparing to (3.9), we see that  $p_{n',3}$  is continuous across the light cone given by  $\kappa = 1$ . It is not smooth, however. Indeed, we have  $p_{n',3}(\xi) = 2C_{n',3}|\xi'|^{-1}G(\kappa)$ , where

$$p_{n',3}(\xi) = 2C_{n',3}|\xi'|^{-1} \begin{cases} G(\pi/2) & \text{if } \kappa \leq 1, \\ \kappa G(\arcsin(1/\kappa)) & \text{if } \kappa > 1, \end{cases}$$

where  $G(s) := \int_0^s \cos^{n'-2} \theta d\theta$ . The function  $\arcsin(1/\kappa)$  can be regularized near  $\kappa = 1$  with the substitution  $\kappa = 1 + t^2$  with  $t$  near 0. We have that  $t \mapsto \arcsin(1/(1+t^2))$  is smooth for  $t \geq 0$  up to  $t = 0$ , and has a non-trivial Taylor expansion at  $t = 0$  valid for  $t \geq 0$ . In fact, its derivative is  $-2(1+t^2)^{-1}(2+t^2)^{-1/2}$  which has even an analytic extension to  $t < 0$ . Then  $\kappa G(\arcsin(1/\kappa))$  has a non-trivial (meaning non-constant) expansion in terms of  $\sqrt{\kappa-1}$  near  $\kappa = 1$ . This shows that  $p_{n',3}$  is smooth all the way to the light cone on both sides of it but not smooth across it. In particular, it has a conormal singularity there.

The dependence on the parity of  $n'$  becomes clear by looking at (3.10). Evaluating the last integral for particular values of  $n'$ , we obtain the following. When  $n' = 3$ ,

$$p_{3,3}(\xi) = 2C_{3,3}|\xi'|^{-1}, \quad |\xi'| > |\xi''|$$

as above. When  $n' = 4$ ,

$$p_{4,3}(\xi) = 2C_{4,3}|\xi'|^{-1} \left( \frac{1}{2} \frac{\sqrt{\kappa^2 - 1}}{\kappa} + \frac{\kappa}{2} \arcsin \frac{1}{\kappa} \right), \quad \kappa \geq 1.$$

When  $n' = 5$ ,

$$p_{5,3}(\xi) = 2C_{5,3}|\xi'|^{-1} \left( 1 - \frac{1}{3\kappa^2} \right), \quad \kappa \geq 1.$$

One can show that the square roots (which are a part of the expansion of  $\arcsin(1/\kappa)$  as well) appear for  $n'$  even only.

#### 4. PROPERTIES OF $p(\xi)$ . INVERSION AND STABILITY

**Theorem 4.1.**  *$L'L$  is a Fourier multiplier with  $p(\xi)$  given by (3.5), which also can be expressed as the integral (3.7). For  $p(\xi)$  we have*

(a)  $p \in L^1_{\text{loc}}(\mathbb{R}^n)$ ; it is smooth away from the light cone, and it is positively homogeneous of order  $-1$ .

(b) If  $n' \geq 3$  or  $n'' \geq 3$ ,  $p$  is continuous across the light cone away from the origin, and

$$(4.1) \quad \frac{1}{C|\xi|} \leq p(\xi) \leq \frac{C}{|\xi|}$$

with some  $C = C(n', n'') > 0$ .

(c) When  $n' = n'' = 2$ , (4.1) holds outside any conic neighborhood of the light cone, i.e., for  $\||\xi'| - |\xi''|\| \geq c_0|\xi|$  with any  $c_0 > 0$ . Inside that set, for  $0 < c_0 \ll 1$ , we have

$$(4.2) \quad \frac{1}{C|\xi|} \leq \frac{p_{2,2}(\xi)}{-\log \frac{\||\xi'| - |\xi''|\|}{|\xi|}} \leq \frac{C}{|\xi|}.$$

*Proof.* We proved the first two parts of (a) above. On the other hand, since the convolution kernel of  $L'L$  is positively homogeneous of order  $-n+1$ ,  $p$  must be positively homogeneous of order  $-1$ .

Consider (b) now. Since  $p \in L^1_{\text{loc}}$  is smooth away from the light cone, it is enough to consider the behavior of  $p$  as  $\xi$  tends to the latter. The integral in (3.7) is absolutely convergent and defines a locally bounded (and, as we prove below, a continuous) function of  $\xi = (\xi', \xi'') \neq 0$  near the light cone (and hence everywhere away from the origin). Also,  $p(\xi) > 0$  for  $\xi' \neq 0$  and  $\xi'' \neq 0$  but representation (3.7) is not convenient to establish positivity when one of those factors vanishes. In that case, say when  $\xi' = 0$ , but  $\xi'' \neq 0$ , we use (3.5) to write

$$\begin{aligned} p(0, \xi'') &= 2\pi |\mathbb{S}^{n'-1}| \int_{\mathbb{S}^{n''-1}} \delta(\theta'' \cdot \xi'') \, dS(\theta'') \\ &= 2\pi |\mathbb{S}^{n'-1}| |\mathbb{S}^{n''-2}| \int_{\mathbb{R}} \delta(|\xi''|t) (1-t^2)_+^{(n''-3)/2} \, dt \\ &= 2\pi |\mathbb{S}^{n'-1}| |\mathbb{S}^{n''-2}| \frac{1}{|\xi''|}. \end{aligned}$$

This proves estimate (3.8) in case (b).

To prove the continuity of  $p$  away from the origin in case (b), and to treat case (c), we derive an “explicit” formula for  $p$  using hypogeometric functions. For now, we consider the general case  $n' \geq 2$  and  $n'' \geq 2$ . Let  $0 < |\xi'| < |\xi''|$ . Set

$$\kappa = \frac{|\xi'|}{|\xi''|}$$

when  $\xi'' \neq 0$ . We have, using the even-ness of the integrand in (3.7), and the substitution  $s = |\xi'|u$ ,

$$\begin{aligned} p(\xi) &= 2\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi'|^{2-n'} |\xi''|^{2-n''} \int_{\mathbb{R}} (|\xi'|^2 - u^2)_+^{(n'-3)/2} (|\xi''|^2 - u^2)_+^{(n''-3)/2} \, du \\ &= 4\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi'|^{2-n'} |\xi''|^{2-n''} \int_0^{|\xi'|} (|\xi'|^2 - u^2)_+^{(n'-3)/2} (|\xi''|^2 - u^2)_+^{(n''-3)/2} \, du \\ &= 4\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi''|^{-1} \int_0^1 (1-s^2)^{(n'-3)/2} (1-\kappa^2 s^2)^{(n''-3)/2} \, ds. \end{aligned}$$

The substitution  $t = s^2$  further yields

$$\begin{aligned} p(\xi) &= 2\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi''|^{-1} \int_0^1 t^{-1/2} (1-t)^{(n'-3)/2} (1-\kappa^2 t)^{(n''-3)/2} \, dt \\ &= 2\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi''|^{-1} \mathcal{B}\left(\frac{1}{2}, \frac{n'-1}{2}\right) {}_2F_1\left(\frac{3-n''}{2}, \frac{1}{2}; \frac{n'}{2}; \kappa^2\right), \end{aligned}$$

where in the last line, we used the Euler’s Integral Representation of the Hypergeometric function ([1, Th 2.2.1]).

Since for the coefficients  $a, b, c$ , see Proposition 4.5 below, we have  $c-a-b = (n'+n''-4)/2 > 1/2$  in case (b), we can take the limit  $\kappa \rightarrow 1-$  to get

$$\begin{aligned} \lim_{\kappa \rightarrow 1-} p(\xi) &= 2\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi''|^{-1} \mathcal{B}\left(\frac{1}{2}, \frac{n'-1}{2}\right) \frac{\Gamma(n'/2) \Gamma((n'+n''-4)/2)}{\Gamma((n'+n''-3)/2) \Gamma((n'-1)/2)} \\ &= 2\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi''|^{-1} \frac{\Gamma(1/2) \Gamma((n-4)/2)}{\Gamma((n-3)/2)} \\ &= 2\pi |\mathbb{S}^{n'-2}| |\mathbb{S}^{n''-2}| |\xi''|^{-1} \mathcal{B}\left(\frac{1}{2}, \frac{n-4}{2}\right). \end{aligned}$$



This expression is symmetric with respect to  $|\xi'|$  and  $|\xi''|$  (note that we can replace  $|\xi''|$  and  $|\xi'|$  when  $\kappa = 1$ ), therefore, the limit  $\kappa \rightarrow 1+$  would yield the same result. Therefore,  $p$  is continuous across the light cone away from the origin.

We proceed with the proof of (c) now. We have  $n' = n'' = 2$ . For  $\kappa \in (0, 1)$ , we have

$$p_{2,2}(\xi) = 8\pi^2 |\xi''|^{-1} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \kappa^2\right),$$

which is also  $16\pi |\xi''|^{-1}$  times the complete elliptic integral of the first kind. Therefore, we have

$$\lim_{\kappa \rightarrow 1-} \frac{p_{2,2}(\xi)}{-\log(1 - \kappa^2)} = 8\pi |\xi''|^{-1}.$$

For  $\kappa \in (1, \infty)$ , the expression for  $p$  is

$$p_{2,2}(\xi) = 8\pi^2 |\xi'|^{-1} {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \kappa^{-2}\right).$$

As  $\kappa \rightarrow 1+$ ,  $\kappa^{-2} \rightarrow 1-$  and we get

$$\lim_{\kappa \rightarrow 1+} \frac{p_{2,2}(\xi)}{-\log(1 - \kappa^{-2})} = 8\pi |\xi'|^{-1}.$$

Note that  $\log(1 - \kappa^{-2}) = \log \kappa^{-2} + \log |1 - \kappa^2| = \log |1 - \kappa^2| + o(1) = \log |1 - \kappa| + O(1)$ , as  $\kappa \rightarrow 1$ . Also,  $\log |1 - \kappa| = \log \frac{||\xi'| - |\xi''||}{|\xi|} = \log \frac{||\xi'| - |\xi''||}{|\xi|} + O(1)$  for  $\kappa$  close to 1. Therefore,

$$(4.3) \quad \lim_{\kappa \rightarrow 1} \frac{|\xi| p_{2,2}(\xi)}{-\log \frac{||\xi'| - |\xi''||}{|\xi|}} = 8\pi.$$

This implies estimate (4.2). □

*Remark 4.2.* In case (b),  $p(\xi)$  is continuous across the light cone but it is not smooth there, by [8, Theorem 7.1.18].

**Corollary 4.3.** *The following inversion formula takes place: for every  $f \in L^2(\Omega)$ , we have*

$$f = p(D)^{-1} L' L f.$$

To formulate a stability estimate, in the case  $n' = n'' = 2$  we need a modified Sobolev space. To this end, define the norm

$$\|h\|_{H_{2,2}^1(\mathbb{R}^4)}^2 := \int_{\mathbb{R}^4} \sigma^2(\xi) |\hat{h}(\xi)|^2 d\xi, \quad \sigma(\xi) := \frac{\langle \xi \rangle}{-\log \frac{||\xi'| - |\xi''||}{e|\xi|}},$$

where  $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ . Since  $||\xi'| - |\xi''|| \leq |\xi|$ , the denominator in  $\sigma$  has a lower bound 1, therefore  $0 \leq \sigma(\xi) \leq \langle \xi \rangle$  but  $\sigma$  is not elliptic. We define the space  $H_{2,2}^1(\mathbb{R}^4)$  as the completion of  $C_0^\infty(\mathbb{R}^4)$  under that norm. It is a Hilbert space.

**Theorem 4.4.**

(a) *When  $n' = n'' = 2$ , we have*

$$(4.4) \quad \|f\|_{L^2(\Omega)/C} \leq \|L' L f\|_{H_{2,2}^1(\mathbb{R}^4)} \leq C \|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega).$$

(b) *When either  $n' \geq 3$  or  $n'' \geq 3$ , we have*

$$(4.5) \quad \|f\|_{L^2(\Omega)/C} \leq \|L' L f\|_{H^1(\mathbb{R}^n)} \leq C \|f\|_{L^2(\Omega)}, \quad \forall f \in L^2(\Omega).$$

*Proof.* We start with (4.4). We take Fourier transforms of  $f$  and  $L'Lf$ . Then the first inequality follows from  $p(\xi)\sigma(\xi) \geq 1/C$ , which is true in a conic neighborhood of the light cone by (4.3), and outside it in a trivial way. For the second inequality in (4.4), we can use  $p(\xi)\sigma(\xi) \leq C$  for  $|\xi| \geq 1$ . In the ball  $|\xi| \leq 1$  however,  $p$  has a singularity at the origin, while  $\sigma$  does not compensate for it; it compensates for the logarithmic factor only. To deal with the singularity at  $\xi = 0$ , we proceed as in the proof of [13, Theorem II.5.1]. Denote by  $q(\xi)$  the function in (4.3), before we take the limit there. It is positively homogeneous of order 0, and has positive lower and upper bounds. We have  $p\sigma = q\langle\xi\rangle/|\xi|$ , and then  $1/C \leq |\xi|p\sigma \leq C$ . Then cutting  $\widehat{L'Lf}$  to  $|\xi| \leq 1$ , we are left with estimating

$$\int_{|\xi| \leq 1} |\xi|^{-2} |\hat{f}(\xi)|^2 d\xi$$

which can be bounded by  $C_s \|f\|_{H^s}^2$  for every  $s$ , see the proof of [13, Theorem II.5.1] as a consequence of the fact that  $|\xi|^{-2}$  is locally integrable, and that  $f$  is compactly supported.

The proof of (b) is similar, except that we have  $\sigma(\xi) = \langle\xi\rangle$  then.  $\square$

We collect some facts about the hypergeometric function

$$(4.6) \quad {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where  $(a)_k := a(a+1)\dots(a+k-1)$  when  $k \geq 1$ , and  $(a)_0 = 1$ , in the following see [1, Theorems 2.1.3, 2.2.2].

**Proposition 4.5.** [1] *The series (4.6) converges absolutely when  $|z| < 1$  and  $c \notin \{0, -1, -2, \dots\}$ . It has the following behavior as  $z \rightarrow 1-$ .*

- If  $\Re(c - a - b) > 0$ , then

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}.$$

*In particular,  ${}_2F_1(a, b; c; z)$  is it continuous up to  $z = 1$  from the left by Abel's theorem.*

- If  $c = a + b$ , then

$$\lim_{z \rightarrow 1-} \frac{{}_2F_1(a, b; c; z)}{-\log(1 - z)} = \frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)}.$$

- If  $\Re(c - a - b) = 0$  and  $c \neq a + b$ , then

$$\lim_{z \rightarrow 1-} (1 - z)^{a+b-c} \left( {}_2F_1(a, b; c; z) - \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} \right) = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}.$$

- If  $\Re(c - a - b) < 0$ , then

$$\lim_{z \rightarrow 1-} \frac{{}_2F_1(a, b; c; z)}{(1 - z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}.$$

## 5. $L$ AS A GENERALIZED RADON TRANSFORM

We will fit  $L$  into the framework of Guillemin's generalized Radon transforms, and in particular, will describe it as a Fourier Integral Operator (FIO). We are not going to get deeper into the analysis and will only hint why  $p$  might be expected to have a non-symbol behavior from an FIO point of view.

This framework starts with picking a connected manifold  $Y$  where our functions are defined, and a connected manifold  $X$  of submanifolds over which we integrate. In our case,  $Y = \mathbb{R}^n$  with  $n_Y = \dim Y = n$ , and we identify the light lines with  $X = \Sigma$  with  $n_X = \dim X = 2n - 3$ , see

(2.2). We choose measures on each one of them, which we already did in the previous sections. The point-line relation  $Z \subset X \times Y = \Sigma \times \mathbb{R}^n$  is given by

$$Z := \{(z, \theta, x) \mid (z, \theta) \in \Sigma, x = z + t\theta \text{ for some } t \in \mathbb{R}\}$$

We parameterize  $Z$  by  $(z, \theta, t) \in \Sigma \times \mathbb{R}$  making it a  $n_Z := 2n - 2$  dimensional connected manifold, and we chose the product measure on it.

Consider the diagram

$$(5.1) \quad \begin{array}{ccc} & Z & \\ \pi_\Sigma \swarrow & & \searrow \pi_{\mathbb{R}^n} \\ \Sigma & & \mathbb{R}^n \end{array}$$

where  $\pi_\Sigma$  and  $\pi_{\mathbb{R}^n}$  are the natural projections. The projection  $\pi_{\mathbb{R}^n}$  is proper, and both are submersions. Then  $Z$  is a double fibration. The transform  $L$  is the generalized Radon transform related to it. Its Schwartz kernel is  $\delta_Z$  with respect to the measure on  $Z$ . As a consequence,  $L \in I^{\frac{1-n}{4}}(\Sigma \times \mathbb{R}^n; N^*Z \setminus 0)$ . Indeed, the order of  $L$  is  $m = k/2 - (n_X + n_Y)/4$ , where  $k = n_X + n_Y - n_Z$  is the codimension of the submanifolds over which we integrate on. We have  $k = n - 1$ , and  $m = (n - 1)/2 - (2n - 3 + n)/4 = (1 - n)/4$ .

The microlocal version of the diagram (5.1) is

$$(5.2) \quad \begin{array}{ccc} & N^*Z \setminus 0 & \\ \pi_{T^*\Sigma} \swarrow & & \searrow \pi_{T^*\mathbb{R}^n} \\ T^*\Sigma \setminus 0 & & T^*\mathbb{R}^n \setminus 0. \end{array}$$

The dimensions in (5.2) from left to right are:  $4n - 6 \geq 3n - 3 \geq 2n$ .

As we found out,  $L'L$  is not a  $\Psi$ DO because  $p(\xi)$  is singular at the light cone. This is an indication that the Bolker condition fails. We will demonstrate this directly. We recall that the Bolker condition requires  $\pi_{T^*\Sigma}$  to be an injective immersion.

We analyze the Lagrangian  $N^*Z$ , and therefore, the canonical relation  $N^*Z' \setminus 0$  in more detail. The operator  $L$  has a delta type of Schwartz kernel with a wavefront set conormal to  $Z$ , i.e., it is a conormal distribution with

$$N^*Z = \left\{ ((z, \theta, x), (\hat{z}, \hat{\theta}, \xi)) \mid x = z + t\theta, \xi = -\hat{z}, \right. \\ \left. \hat{\theta} = t(-\xi + (\xi \cdot \theta)\theta) \right\},$$

where hats indicate dual variables,  $\hat{\theta}$  is considered as conormal on  $\mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$  at  $\theta$ , and  $\hat{z}$  is conormal to  $\theta$ . Thus, parameters on  $N^*Z$  are chosen to be  $(z, \theta, t, \xi)$  with  $\theta \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$ ,  $z \perp \theta$ ,  $\xi \perp \theta$ ,  $t \in \mathbb{R}$ . The canonical relation associated to  $L$  is given by

$$(5.3) \quad C := N^*Z' \setminus 0 = \left\{ ((z, \theta, \hat{z}, \hat{\theta}), (x, \xi)) \in T^*(\Sigma \times \mathbb{R}^n) \setminus 0 \mid \right. \\ \left. x = z + t\theta, \hat{z} = \xi, \hat{\theta} = t(\xi - (\xi \cdot \theta)\theta) \right\}.$$

We rearranged the variables in accordance with the standard convention. We can check directly that  $(\hat{z}, \hat{\theta}) = 0$  if and only if  $\xi = 0$ . As a consequence,

$$\text{WF}(Lf) \subset C \circ \text{WF}(f)$$

as it follows from the Hörmander-Sato lemma. We have

$$\pi_{T^*\Sigma}((z, \theta, \hat{z}, \hat{\theta}), (x, \xi)) = (z, \theta, \hat{z}, \hat{\theta}) = (z, \theta, \xi, t(\xi - (\xi \cdot \theta)\theta)),$$

written in the parameterization  $(z, \theta, t, \xi)$ . Then injectivity of  $\pi_{T^*\Sigma}$  is equivalent to unique solvability of  $t(\xi - (\xi \cdot \theta)\theta) = \hat{\theta}$  with respect to  $t$  given  $(z, \theta, \xi)$ . Since  $\hat{\theta} \perp \theta$ , this equation is solvable, uniquely, when  $\xi$  is not collinear to  $\theta$ . When  $\xi \parallel \theta$ , uniqueness fails when it is solvable, i.e., when  $\hat{\theta} = 0$ . This is a failure of the Bolker condition.

We consider the projection

$$\pi_{T^*\mathbb{R}^n}((z, \theta, \hat{z}, \hat{\theta}), (x, \xi)) = (x, \xi) = (z + t\theta, \xi)$$

briefly as well. Recall that  $(z, \theta, t, \xi)$  are parameters here with  $z$  and  $\xi$  perpendicular to  $\theta \in \mathbb{S}^{n'-1} \times \mathbb{S}^{n''-1}$ . This projection is surjective because given  $\xi \in \mathbb{R}^n \setminus 0$ , we can always find  $\theta$  as above normal to it, as we saw in the proof of Corollary 2.2. This is in contrast to the Minkowski case (either  $n' = 1$  or  $n'' = 1$ ), where the range of  $\pi_{T^*\mathbb{R}^n}$  is the causal cone only since the equation  $\xi \cdot (1, \theta) = 0$  with  $|\theta| = 1$  has a solution when  $|\xi'| \leq |\xi''|$  only, say when  $n' = 1$ . For this reason, we can resolve all singularities in the case  $n' \geq 2$ ,  $n'' \geq 2$  under consideration, while we can do that only on the spacelike cone in the Minkowski case.

We formulate the the properties we established below.

**Theorem 5.1.**

- (a)  $L \in I^{\frac{1-n}{4}}(\Sigma \times \mathbb{R}^n; N^*Z \setminus 0)$  with  $\Sigma$  as in (2.2) and canonical relation given by (5.3).
- (b) The Bolker condition is satisfied if and only if  $|\xi'| \neq |\xi''|$ .

The theorem allows us to use pseudo-differential cutoffs either near the light cone, or near the image of it under  $C$ , as it is done in [10, 11] in the Lorentzian case to get recovery of not-lightlike singularities, say with partial measurements, or when the metric is not flat. The treatment of lightlike singularities is more delicate and requires special classes of FIOs corresponding to two cleanly intersection Lagrangians, see [3–5]. This is done in [21] for the light ray transform in Lorentzian geometry. We do not pursue this direction in the present paper.

## 6. COMPARISON TO THE LIGHT RAY TRANSFORM IN MINKOWSKI SPACETIME

Throughout this paper, we assumed  $n' \geq 2$ ,  $n'' \geq 2$ . The case of one of them being 1 is the Minkowski case studied previously, as explained in the introduction. We review briefly some of the results for that case to compare them to the rest of the paper, see also [11, 17, 18].

In the Minkowski case, we routinely call the dimension  $1 + d$  (not  $n$  or  $d!$ ), i.e.,  $(t, x) \in \mathbb{R}^{1+d}$ . Then  $n' = 1$ ,  $n'' = d$ , and  $n = 1 + d$ . We also denote the dual variable by  $(\tau, \xi)$ . The light ray transform is defined as

$$Lf(z, \theta) = \int_{\mathbb{R}} f(s, z + s\theta) ds, \quad (z, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1}.$$

Note that we do not have  $(0, z) \perp (1, \theta)$  (but we could have chosen the initial point to be on  $(1, \theta)^\perp$ ), and we integrate over future pointing light like rays, i.e., those in the direction of  $(1, \theta)$ , omitting  $(-1, \theta)$  because that leads to the same kind of integral. This leads to some mismatch of the constants if we compare the formulas for  $L'L$  in the previous sections when formally  $n' = 1$  there. Then  $L'L$  is a convolution with

$$\frac{\delta(t - |x|) + \delta(t + |x|)}{|x|^{d-1}},$$

compare with Theorem 3.3. We have

$$p(\xi) = 2\pi |\mathbb{S}^{d-2}| \frac{(|\xi|^2 - \tau^2)_+^{\frac{d-3}{2}}}{|\xi|^{d-2}}.$$

This replaces (3.7) in this case. The multiplier  $p$  is still positively homogeneous of degree  $-1$ , non-negative but it vanishes identically for  $|\xi| < |\tau|$ , i.e., in the timelike cone in  $T^*\mathbb{R}^{1+d}$ . It is singular at the lightlike cone. As a result,  $\hat{f}(\xi)$  in the timelike cone cannot be recovered stably, and timelike singularities with localized measurements are lost. This is in contrast to the case  $n' \geq 2$ ,  $n'' \geq 2$  we study in this paper.

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