Travel Time Tomography and Tensor Tomography, I

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Alternative titles:

Boundary/Lens Rigidity
Inverse Kinematic Problem
Integral Geometry of Tensor Fields

References

Most references available at my web page.

Boundary rigidity and tensor tomography for simple metrics:

- Stability estimates for the X-ray transform of tensor fields and boundary rigidity (with Gunther Uhlmann), *Duke Math. J.* 123(2004), 445–467.
- Boundary rigidity and stability for generic simple metrics (with Gunther Uhlmann), *Journal Amer. Math. Soc.* 18(2005), 975–1003.

Lens rigidity and tensor tomography with incomplete data for a class of non-simple metrics:

- Integral geometry of tensor fields on a class of non-simple Riemannian manifolds (with Gunther Uhlmann), *Amer. J. Math.* 130(1)(2008), 239–268.
- Local lens rigidity with incomplete data for a class of non-simple Riemannian manifolds (with Gunther Uhlmann), *J. Diff. Geom.* 82(2) (2009), 383–409.

More references

A survey of those papers, with many technical details sketched only:

- Microlocal Approach to Tensor Tomography and Boundary and Lens Rigidity, *Serdica Math. J.* 34(1)(2008), 67–112.

If you are going to read only one of those papers, this is the one.

Integral geometry of functions for arbitrary curves:

- The X-Ray transform for a generic family of curves and weights, with Bela Frigyik and Gunther Uhlmann, *J. Geom. Anal.* 18(1)(2008), 8197.

A philosophical paper:

- Linearizing non-linear inverse problems and an application to inverse backscattering (with Gunther Uhlmann), *J. Funct. Anal.* 256(9)(2009), 2842–2866.

Excellent Lecture Notes

- Vladimir Sharafutdinov, Ray Transform on Riemannian manifolds.

Lecture 1

In this lecture, we formulate the problem and motivate our interest in it.

- Travel Time Seismology as a motivating example
- Boundary Rigidity, definition
- Lens Rigidity, definition
- Linearizing the Boundary Rigidity Problem: Tensor Tomography

Travel Time Seismology

To image the inner structure of the Earth, we need signals that can get from there to the surface. One such signal (and perhaps the only usable one) are seismic waves. Each time there is an earthquake, a network of seismic stations around the world record the seismic wave that arrives there and in particular, time it takes the wave to get there. The speed of those waves depends on the structure of the Earth, and one hopes to use this information to recover the latter.

A good model is a domain (a ball) in $\mathbf{R}^3$ with a Riemannian metric g in it. More about this later.

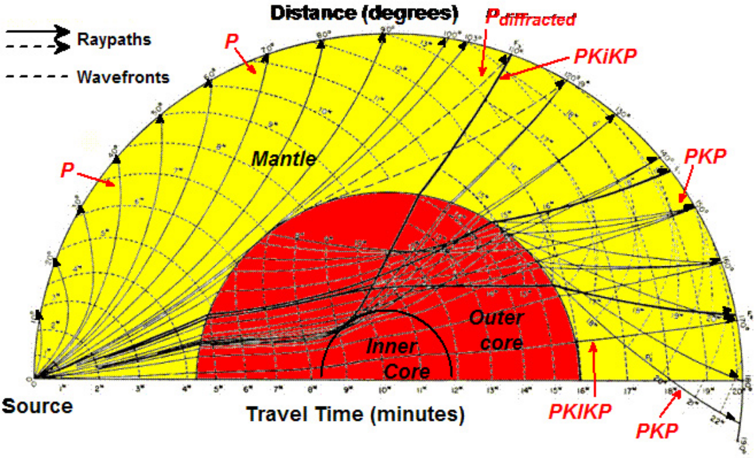


Figure: Travel times of P-waves through Earth. Picture taken from the web page of L. Braile, Purdue University.

Travel Time Seismology

This problem was first studied in the beginning of the 20th century by **Herglotz**, and **Wiechert & Zoeppritz** in an attempt to recover the inner structure of the Earth from travel times of seismic waves. They solved explicitly a partial case of this problem: when M (the Earth) is a ball, and g is a radially symmetric isotropic metric, i.e.,

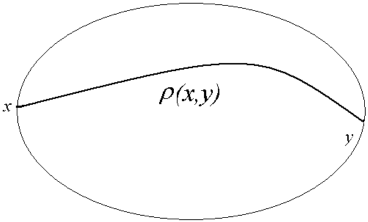
$$ds^2 = a^2(r)dx^2, \quad r := |x|.$$

They imposed an assumption that there are no multiple arrival times (simplicity assumption) as well. Travel time seismology is still one of the main methods to study the inner structure of the Earth today.

Other possible applications: in medical imaging.

The Boundary Rigidity Problem (Travel Time Tomography)

Let M be a compact domain (manifold) with boundary. Let $g = \{g_{ij}\}$ be a Riemannian metric on M. Let $\rho(x, y)$ be the distance between any two boundary points x, y (in the metric g).



Boundary Rigidity: *Does ρ, known on $\partial M \times \partial M$, determine uniquely g?*

The distance $\rho(x, y)$ is equal to the travel time of the *unique* signal coming from x and measured at y under the following *simplicity conditions*:

- $\forall (x, y) \in \partial M \times \partial M$, there is unique geodesics connecting x, y, depending smoothly on (x, y) (i.e., no caustics);
- ∂M is strictly convex.

Then we call g a *simple* metric.

The answer is negative because for every diffeomorphism ψ fixing ∂M pointwise, the metric $\psi_* g$ has the same data as g! Here,

$$(\psi_* g)_{ij} = g_{kl} \frac{\partial \psi^k}{\partial x^i} \frac{\partial \psi^l}{\partial x^j},$$

and we use the Einstein summation convention.

So the right question to ask is whether we can recover g, up to an isometry as above, from the boundary distance function.

In other words, if $\rho_{\hat{g}} = \rho_g$, is there a diffeo $\psi : M \to M$, $\psi|_{\partial M} = Id$, such that $\psi_* \hat{g} = g$?

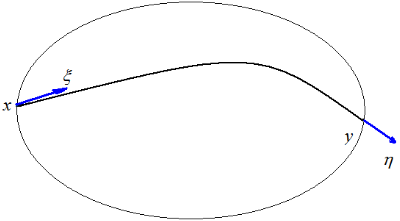
Assume that g is isotropic, i.e., $g_{ij}(x) = c(x)\delta_{ij}$. Physically, this corresponds to a variable wave speed that does not depend on the direction of propagation. Then any ψ that is not identity (but is identity on ∂M), will make g anisotropic. Therefore, in the class of the isotropic metrics, we do not have the freedom to apply isometries and we would expect g to be uniquely determined. This is known to be true for simple metrics (Mukhometov, Romanov, et al.)

Our interest is in anisotropic metrics.

If the metric is not simple, the answer is negative in general. A “slow region” inside M cannot be seen from the distance function.

Multiple Arrival Times: The Lens Rigidity Problem (The Inverse Kinematic Problem)

Define the scattering relation σ and the length (travel time) function ℓ:



$$\sigma : (x, \xi) \to (y, \eta), \quad \ell(x, \xi) \to [0, \infty].$$

Diffeomorphisms preserving ∂M pointwise do not change σ, ℓ!

Lens rigidity: *Do σ, ℓ determine uniquely g, up to an isometry?*

In other words, if $\sigma_{\hat{g}} = \sigma_g$, $\ell_{\hat{g}} = \ell_g$, is there a diffeo $\psi : M \to M$, $\psi|_{\partial M} = Id$, such that $\psi_* \hat{g} = g$?

No, in general but the counterexamples are harder to construct.

The lens rigidity problem and the boundary rigidity one are equivalent for simple metrics! Indeed, then $\rho(x, y)$, known for x, y on ∂M determines σ, ℓ uniquely, and vice-versa.

Exercise: Prove it!

Hint: Let's assume that we know $\rho|_{\partial M \times \partial M}$. Take the tangential gradient $\operatorname{grad}'_x \rho(x, y)$ and $\operatorname{grad}'_y \rho(x, y)$. Prove that for the full gradients we have

$$\operatorname{grad}_x \rho(x, y) = -\xi, \quad \operatorname{grad}_y \rho(x, y) = \eta,$$

with ξ, η as above and unit. Now, since they are unit, they are uniquely determined by their projections ξ', η' that we know.

If we know σ, ℓ then we can integrate along appropriate curves to reconstruct $\rho|_{\partial M \times \partial M}$.

For non-simple metrics (caustics and/or non-convex boundary), the Lens Rigidity is the right problem to study.

Even for non-simple metrics, one can still recover σ, ℓ from the travel times, but we need multiple arrival times, and a non-degeneracy assumption.

Linearization of the Boundary Rigidity Problem

Let g^s, $|s| \ll 1$ be an one-parameter family of metrics. Let $\rho_s(x,y)$ be the corresponding distance function. We want to compute $\frac{d\rho_s}{ds}|_{s=0}$.

$$\rho_s(x,y) = \int_0^1 \sqrt{g^s_{ij}(\gamma_s(t))\dot\gamma^i_s(t)\dot\gamma^j_s(t)}\,dt,$$

where $[0,1] \ni t \mapsto \gamma_s$ is the unique geodesic connecting x and y. Notice that the integrand equals $\rho_s(x,y)$ and is independent of t.

Differentiate w.r.t. s at $s=0$. The variable s occurs 4 times. The (combined) derivative w.r.t. the red ones is zero because for a fixed metric, the geodesics minimize the length functional! So only the blue derivative survives.

So we get

$$\frac{d\rho_s}{ds}\Big|_{s=0} = \frac{1}{2\rho_0}\int_0^1 f_{ij}(\gamma(t))\dot\gamma^i(t)\dot\gamma^j(t)\,dt, \quad f := \frac{dg^s}{ds}\Big|_{t=0}.$$

Change the parameterization of the geodesics from constant speed to unit speed to get

$$\frac{d\rho_s}{ds}\Big|_{s=0} = \frac{1}{2}\int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,dt, \quad f := \frac{dg^s}{ds}\Big|_{s=0},$$

where the integral is taken over the unit speed geodesic connecting x and y.

If we have two metrics g and $\hat{g}$, close to each other, we can set

$$g^s = s\hat{g} + (1-s)g, \quad 0 \le s \le 1.$$

Then

$$f = \hat{g} - g,$$

and

$$\frac{d\rho_s}{ds}\Big|_{s=0} \approx \hat{\rho} - \rho.$$

Linearized problem (Tensor Tomography)

Recover a tensor field f_{ij} from the geodesic X-ray transform

$$I_g f(\gamma) = \int f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t)\,dt$$

known for all (or some) max geodesics γ in M.

This problem should not have a unique solution because it linearizes one that has no unique solution (remember the diffeomorphism).

Given a vector field v, take a diffeomorphism

$$\psi(x) = x + \epsilon v(x) + O(\epsilon^2).$$

Then g and $\psi_* g$ have the same data. One can check that

$$\psi_* g = g + 2\epsilon dv + O(\epsilon^2).$$

where dv is the symmetric differential of v defined by

$$[dv]_{ij} = (\nabla_i v_j + \nabla_j v_i)/2, \quad \text{here } \nabla = \text{covariant derivative.}$$

Next, $v|_{\partial M} = 0$ because $\psi|_{\partial M} = Id$.

So we arrive at

$$I_g(dv) = 0$$

for any such v.

This is not hard to check directly. Indeed,

$$\big([dv]_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^i(t)\big) = \frac{d}{dt}\langle v(\gamma(t)), \dot{\gamma}(t)\rangle.$$

Integrate w.r.t. t; then the l.h.s. is $I_g(dv)$, while the r.h.s. vanishes by the Fundamental Theorem of Calculus (because $v = 0$ on ∂M).

We expect this to be the whole kernel, which justifies the following.

Definition 1

We say that I_g is s-injective, if $I_g f = 0$ implies $f = dv$ with $v|_{\partial M} = 0$.

So the linearized version of the boundary rigidity problem is to show that I_g is s-injective for simple g.

This is still an open problem. So far we know that this is true for a dense and open set of simple metrics.