Introduction to rough paths



 $\dot{y}_t = b(y_t)dt + \sigma(y_t)\dot{B}_t$

where

regular enough coeff 6,5 B random forcing

Important variant Stochastic PDES like

 $\partial_t u_t(x) = \Delta u_t(x) + \sigma(u) \dot{w}_t(x)$

Brownian motion Goussian process s.t.

 $B_3 = 0$, $B_t \sim \mathcal{N}(0, t)$,

{Btitin, j=1,..,n} independent

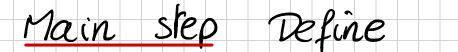
> Easy generalization to (B',.., B°)

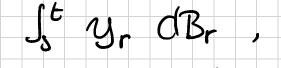
Problem with B Not differentiable

$B \in C^{\frac{1}{2}-\varepsilon}(D,T), \forall T$

Integral equation Recast de as

 $y_{e} = a_{+} \int^{t} b(y_{r}) dr + \int^{t} \sigma(y_{r}) dB_{r}$ $\delta y_{st} = \int_{S} t b(y_r) dr + \int_{S}^{t} \sigma(y_r) dB_r$





for a reasonable class of processes y.

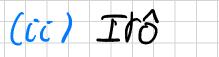
(1936)

(1943)

(1998)

Aim Talk about 3 settings

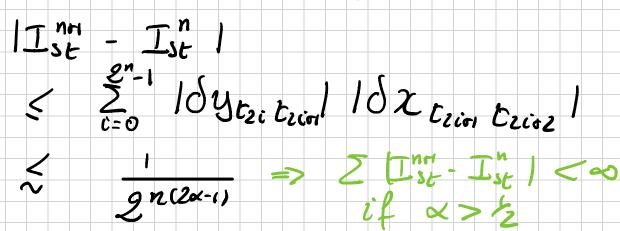




(iii) Raugh paths

2) Young integration <u>Aim</u> consider a Hölder on CO,TJ $x, y \in C^{\alpha}$, with $\alpha > \frac{1}{2}$ we wish to define $= I_{se}^{n}$ $\int_{S}^{t} y_{r} dx_{r} = \lim_{n \to \infty} \sum y_{t_{i}} \delta x_{c_{i}t_{i}r_{i}}$ where $t_i = t_i^n = \frac{(t-s)}{2^n} i + s$ strategy Define $\int_{S}^{t} y_{r} dz_{r} = I_{st}^{o} + Z_{n=0}^{o} (I_{st}^{n+1} - I_{st}^{n})$ Tecomposition Write $= \sum_{i=0}^{n} \mathcal{Y}_{t_{i}}^{n_{ri}} \left(\delta \chi_{t_{i}}^{n_{ri}}, \frac{n_{ri}}{t_{i}} + \delta \chi_{t_{i}}^{n_{ri}}, \frac{n_{ri}}{t_{i}} \right)$ Tst = 2 yenn Sx criticite + yenn Sx criticite $= \sum \frac{1}{3t} - \frac{1}{3t} + \frac{1}{2t}$ $= \sum_{c=0}^{2} Syt_{2i} t_{2in} SZ t_{2in} t_{2io2}$

Bounds Recall: 2, y EC×



Conclusion For x > 2,

Jt yr dr defined if xy EC~

3 Its integration

Hyp oys depends "on the past"

• x = Brownian motion

 \Rightarrow $y_s \perp \delta x_{st}$ if s < t

Bound

 $\mathbb{E}\left\{|\mathbf{I}_{st}^{n_{H}}-\mathbf{I}_{st}^{n}|^{2}\right\}$ $= \sum_{i,j=0}^{2^{n}-1} E_{1}^{j} Sy_{t_{2i},t_{2in}} S \chi_{t_{2in},t_{2in}}$ Sytes tion SX tein times $= \sum_{i=0}^{2^{n}} E \left(\left(\underbrace{Sy_{ei} t_{iin}}_{iin} \right)^{2} \left(\underbrace{SX} \right)_{t_{2in}}^{2} \right)$ $= \sum_{i=1}^{2^{n}-1} E\left[\left(\Im_{t_{2i}t_{uin}}\right)^{2}\right] E\left[\left(\Im_{t_{2i}t_{uin}}\right)^{2}\right]$ Rmk If o x BM oy = f(x) or $dy = \sigma(y) dx$ Then $E \left[\left(S \chi_{\text{tein tuise}} \right)^2 \right] \stackrel{\sim}{\prec} \frac{1}{2^n}$ $\mathcal{E}\left[\left(\bigcup_{t_{2i}} t_{u_{H}}\right)^{2}\right] \stackrel{2}{\prec} \frac{1}{9^{n}}$

conclusion we have

 $\mathbb{E}\left\{|\mathcal{I}_{st}^{n_{4}}-\mathcal{I}_{st}^{n}|^{2}\right\}\lesssim\frac{1}{2^{n_{4}}}$

=> Ist converges as in Young case

Bmk

• This convergence is in 22(2)

o It relies on a probabilistic structure

(4) Rough paths setting for x we assume (2) $x \in C^{\alpha}$ with $\frac{1}{2} \le \alpha < \frac{1}{3}$

(ic) There exists

 $X_{st}^{2,ij} = \int_{s < r_1 < r_2 < t} d\chi_{r_1}^{i} d\chi_{r_2}^{j}$

and $x \in C^{2\alpha}$

Called rough path above z

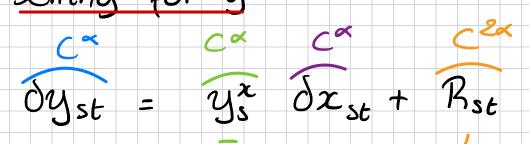


· Existence of z² not automatic

o Has to be proved by probabilishic arg

o otherwise theory is deterministic

setting for y



" dy " Remainder, more regular

Heuristics we have

Jyr dr = ys Srst + / Sysr chr = $y_s \delta x_{st} + \int_{s}^{t} (y_s \delta x_{sr} + R_{sr}) C x_r$ = ys Sist t ys Ist t Jt Rsr chr

~ Young integral

what can we do with RP

· Solve very general noisy systems

· Regularity structures: SPDEs

Signatures & links to algebra,
data analysis