Introduction to rough paths

① Intro

Basic problem in stoch analysis Give meaning to

$$\dot{y}_t = b(y_t)\,dt + \sigma(y_t)\dot{B}_t$$

where

b, σ regular enough coeff

$\dot{B}$ random forcing

Important variant Stochastic PDEs like

$$\partial_t u_t(x) = \Delta u_t(x) + \sigma(u)\,\dot{W}_t(x)$$

Brownian motion Gaussian process s.t.

$B_0 = 0$, $B_t \sim \mathcal{N}(0,t)$,

$\{\delta B_{t_j t_{j+1}} ; j = 1, .., n\}$ independent

↳ Easy generalization to $(B^1, .., B^d)$

Problem with B Not differentiable

$$B \in C^{\frac{1}{2}-\varepsilon}([0,T]), \quad \forall T$$

Integral equation Recast sde as

$$y_t = a + \int_0^t b(y_r)\,dr + \int_0^t \sigma(y_r)\,dB_r$$

$$\delta y_{st} = \int_s^t b(y_r)\,dr + \int_s^t \sigma(y_r)\,dB_r$$

Main step Define

$$\int_s^t y_r\,dB_r ,$$

for a reasonable class of processes y.

Aim Talk about 3 settings

(i) Young (1936)

(ii) Itô (1943)

(iii) Rough paths (1998)

② Young integration

<u>Aim</u> consider

$x, y \in C^\alpha$ (α-Hölder on $[0,T]$), with $\alpha > \frac{1}{2}$

we wish to define

$$\int_s^t y_r \, dx_r = \lim_{n\to\infty} \overbrace{\sum y_{t_i} \, \delta x_{t_i t_{i+1}}}^{\equiv I_{st}^n},$$

where $t_i = t_i^n = \frac{(t-s)}{2^n} i + s$

<u>Strategy</u> Define

$$\int_s^t y_r \, dx_r = I_{st}^0 + \sum_{n=0}^{\infty} \left(I_{st}^{n+1} - I_{st}^n \right)$$

<u>Decomposition</u> Write

$$I_{st}^n = \sum_{i=0}^{2^n-1} y_{t_i^n} \, \delta x_{t_i^n t_{i+1}^n}$$

$$= \sum_{i=0}^{2^n-1} y_{t_{2i}^{n+1}} \left(\delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + \delta x_{t_{2i+1}^{n+1}, t_{2i+2}^{n+1}} \right)$$

$$I_{st}^{n+1} = \sum_{i=0}^{2^n-1} y_{t_{2i}^{n+1}} \, \delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + y_{t_{2i+1}^{n+1}} \, \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}}$$

$$\Rightarrow I_{st}^{n+1} - I_{st}^{n+1} = \sum_{i=0}^{2^n-1} \delta y_{t_{2i} t_{2i+1}} \, \delta x_{t_{2i+1} t_{2i+2}}$$

Bounds Recall : $x, y \in C^\alpha$

$$|I^{n+1}_{st} - I^n_{st}|$$
$$\leq \sum_{i=0}^{2^n-1} |\delta y_{t_{2i} t_{2i+1}}| \, |\delta x_{t_{2i+1} t_{2i+2}}|$$
$$\lesssim \frac{1}{2^{n(2\alpha-1)}}$$

$\Rightarrow \sum |I^{n+1}_{st} - I^n_{st}| < \infty$ if $\alpha > \frac{1}{2}$

Conclusion For $\alpha > \frac{1}{2}$,

$\int_s^t y_r \, dx_r$ defined if $x, y \in C^\alpha$

③ Itô integration

Hyp
- y_s depends "on the past"
- $x \equiv$ Brownian motion

$\Rightarrow$ $y_s \perp\!\!\!\perp \delta x_{st}$ if $s < t$

Bound

$$\mathbb{E}\{|I^{n+1}_{st} - I^{n}_{st}|^2\}$$

$$= \sum_{i,j=0}^{2^n-1} \mathbb{E}\{\delta y_{t_{2i}t_{2i+1}} \ \delta x_{t_{2i+1}t_{2i+2}} \ \delta y_{t_{2j}t_{2j+1}} \ \delta x_{t_{2j+1}t_{2j+2}}\}$$

$$\overset{\perp\!\!\!\perp}{=} \sum_{i=0}^{2^n-1} \mathbb{E}\{(\delta y_{t_{2i}t_{2i+1}})^2 (\delta x)^2_{t_{2i+1}t_{2i+2}}\}$$

$$\overset{\perp\!\!\!\perp}{=} \sum_{i=0}^{2^n-1} \mathbb{E}[(\delta y_{t_{2i}t_{2i+1}})^2] \ \mathbb{E}[(\delta x_{t_{2i+1}t_{2i+2}})^2]$$

Rmk If

- x BM
- $y = f(x)$ or $dy = \sigma(y)\,dx$

Then

$$\mathbb{E}[(\delta y_{t_{2i}t_{2i+1}})^2] \asymp \frac{1}{2^n} \qquad \mathbb{E}[(\delta x_{t_{2i+1}t_{2i+2}})^2] \asymp \frac{1}{2^n}$$

Conclusion we have

$$\mathbb{E}\{|I^{n+1}_{st} - I^{n}_{st}|^2\} \lesssim \frac{1}{2^n}$$

$\Rightarrow I^{n}_{st}$ converges as in Young case

Rmk

- This convergence is in $L^2(\Omega)$
- It relies on a probabilistic structure

④ Rough paths

Setting for x we assume

(i) $x \in C^{\alpha}$ with $\frac{1}{2} \leq \alpha < \frac{1}{3}$

(ii) There exists

$$\mathbb{X}^{2,ij}_{st} = \int_{s<r_1<r_2<t} dx^{i}_{r_1}\, dx^{j}_{r_2}$$

and $x \in C^{2\alpha}$

Called rough path above x

Rmk

- Existence of $\mathbb{X}^2$ not automatic
- Has to be proved by probabilistic arg
- otherwise theory is deterministic

Setting for y

$$\overbrace{\delta y_{st}}^{C^\alpha} = \overbrace{y_s^x}^{C^\alpha} \overbrace{\delta x_{st}}^{C^\alpha} + \overbrace{R_{st}}^{C^{2\alpha}}$$

y_s^x = " $\frac{dy}{dx}$ "

R_{st}: Remainder, more regular

Heuristics We have

$$\int_s^t y_r \, dx_r = y_s \, \delta x_{st} + \int_s^t \delta y_{sr} \, dx_r$$

$$= y_s \delta x_{st} + \int_s^t \left(y_s \, \delta x_{sr} + R_{sr} \right) dx_r$$

$$= y_s \delta x_{st} + y_s \mathbb{X}_{st}^2 + \underbrace{\int_s^t R_{sr} \, dx_r}_{\simeq \text{ Young integral}}$$

What can we do with RP

- Solve very general noisy systems
- Regularity structures: SPDEs
- Signatures & links to algebra,
  data analysis