Intersection and conditioning

Situation:

- Urn with 8 Red and 4 White balls
- Draw 2 balls without replacement

Question: Let

- R_1 = 1st ball drawn is red
- R_2 = 2nd ball drawn is red

Then find $\mathbf{P}(R_1 R_2)$

Intersection and conditioning (2)

Recall:

- Urn with 8 Red and 4 White balls
- Draw 2 balls without replacement

Computation: We have

$$\mathbf{P}(R_1 R_2) = \mathbf{P}(R_1)\mathbf{P}(R_2 | R_1)$$

Thus

$$\mathbf{P}(R_1 R_2) = \frac{8}{12}\frac{7}{11} = \frac{14}{33} \simeq .42$$

8R, 4W

Intersection of 3 events

R_1 = R for 1st draw
R_2 = R for 2nd draw
W_3 = W for 3rd draw

Then

$$\mathbb{P}(W_3 \cap R_2 \cap R_1)$$

$$= \mathbb{P}(R_1)\ \mathbb{P}(R_2 \mid R_1)\ \mathbb{P}(W_3 \mid R_1 \cap R_2)$$

$$= \frac{8}{12} \times \frac{7}{11} \times \frac{4}{10} \approx .17$$

The multiplication rule

For $n=3$: $\mathbb{P}(E_1 E_2 E_3) = \mathbb{P}(E_1)\mathbb{P}(E_2 | E_1)\mathbb{P}(E_3 | E_1 E_2)$

Proposition 2.

Let

- **P** a probability on a sample space S
- $E_1, \ldots, E_n$ n events

Then

$$\mathbf{P}(E_1 \cdots E_n) = \mathbf{P}(E_1) \prod_{k=1}^{n-1} \mathbf{P}(E_{k+1} | E_1 \cdots E_k) \tag{2}$$

Proof

Expression for the rhs of (2):

$$\mathbf{P}(E_1)\,\frac{\mathbf{P}(E_1E_2)}{\mathbf{P}(E_1)}\,\frac{\mathbf{P}(E_1E_2E_3)}{\mathbf{P}(E_1E_2)}\cdots\frac{\mathbf{P}(E_1\cdots E_{n-1}E_n)}{\mathbf{P}(E_1\cdots E_{n-1})}$$

Conclusion:
By telescopic simplification

Example: deck of cards (1)

Situation:

- Ordinary deck of 52 cards
- Division into 4 piles of 13 cards

Question: If

$$E = \{\text{each pile has one ace}\},$$

compute $\mathbf{P}(E)$

Example: deck of cards (2)

Model: Set

$$
\begin{aligned}
E_1 &= \{\text{the ace of S is in any one of the piles}\} \\
E_2 &= \{\text{the ace of S and the ace of H are in different piles}\} \\
E_3 &= \{\text{the aces of S, H \& D are all in different piles}\} \\
E_4 &= \{\text{all 4 aces are in different piles}\}
\end{aligned}
$$

We wish to compute

$$
\mathbf{P}\left(E_1 E_2 E_3 E_4\right)
$$

Example: deck of cards (3)

Applying the multiplication rule: write

$$\mathbf{P}(E_1E_2E_3E_4) = \mathbf{P}(E_1)\,\mathbf{P}(E_2|\,E_1)\,\mathbf{P}(E_3|\,E_1E_2)\,\mathbf{P}(E_4|\,E_1E_2E_3)$$

Computation of $\mathbf{P}(E_1)$: Trivially

$$\mathbf{P}(E_1) = 1$$

Computation of $\mathbf{P}(E_2|\,E_1)$: Given E_1,

- Reduced space is
  $\{51$ labels given to all cards except for ace S$\}$
- $\mathbf{P}(E_2|\,E_1) = \frac{51-12}{51} = \frac{39}{51}$

Example: deck of cards (4)

Other conditioned probabilities:

$$\begin{aligned} \mathbf{P}\left(E_3 \mid E_1 E_2\right) &= \frac{50-24}{50} = \frac{26}{50}, \\ \mathbf{P}\left(E_4 \mid E_1 E_2 E_3\right) &= \frac{49-36}{49} = \frac{13}{49} \end{aligned}$$

Conclusion: We get

$$\begin{aligned} \mathbf{P}(E) &= \mathbf{P}\left(E_1\right) \mathbf{P}\left(E_2 \mid E_1\right) \mathbf{P}\left(E_3 \mid E_1 E_2\right) \mathbf{P}\left(E_4 \mid E_1 E_2 E_3\right) \\ &= \frac{39 \cdot 26 \cdot 13}{51 \cdot 50 \cdot 49} \simeq .105 \end{aligned}$$

Outline

Thomas Bayes

Some facts about Bayes:

- England, 1701-1760
- Presbyterian minister
- Philosopher and statistician
- Wrote 2 books in entire life
- Bayes formula unpublished

Decomposition of $\mathbf{P}(E)$

Proposition 3.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events with $0 < \mathbf{P}(F) < 1$

Then

$$\mathbf{P}(E) = \mathbf{P}(E|F)\,\mathbf{P}(F) + \mathbf{P}(E|F^c)\,\mathbf{P}(F^c)$$

Situation : we know

$\mathbb{P}(E \mid F)$ → we also need $\mathbb{P}(E \mid F^c)$, $\mathbb{P}(F)$

Can we compute

$\mathbb{P}(F \mid E)$?

Bayes' formula

Proposition 4.

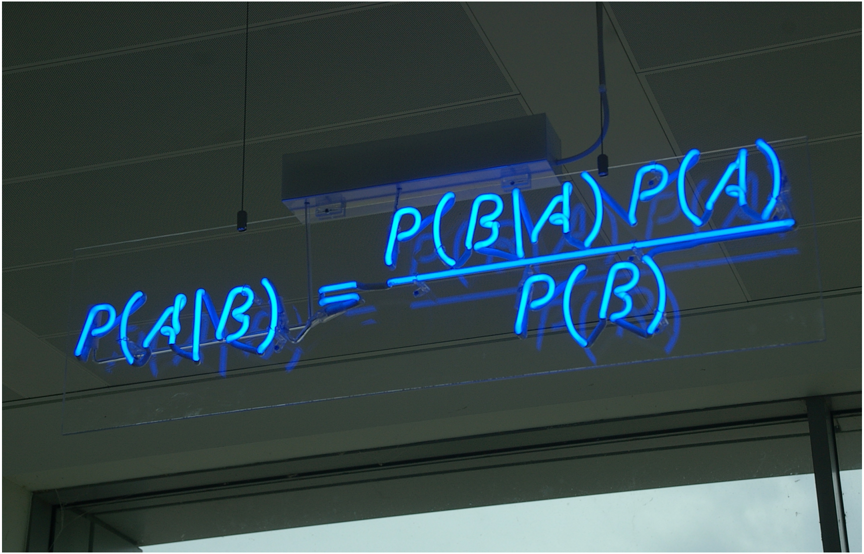
Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events with $0 < \mathbf{P}(F) < 1$

Then

$$\mathbf{P}(F|E) = \frac{\mathbf{P}(E|F)\,\mathbf{P}(F)}{\mathbf{P}(E|F)\,\mathbf{P}(F) + \mathbf{P}(E|F^c)\,\mathbf{P}(F^c)}$$

Iconic Bayes (offices of HP Autonomy)



Example: insurance company (1)

Situation:

- Two classes of people:
  those who are accident prone and those who are not.
- Accident prone: probability .4 of accident in a one-year period
- Not accident prone: probab .2 of accident in a one-year period
- 30% of population is accident prone

Question:
Probability that a new policyholder will have an accident within a year of purchasing a policy?

A: accident prone A_1: accident in 1 year

Insurance example

Sample space: for our experiment

$$S = \{(A, A_1); (A, A_1^c); (A^c, A_1); (A^c, A_1^c)\}$$

Probab: defined through conditioning.

A: accident prone A_1: accident in 1 year

Data: $\mathbb{P}(A_1 \mid A) = .4$

$\mathbb{P}(A_1 \mid A^c) = .2$

$\mathbb{P}(A) = .3$ $\mathbb{P}(A^c) = .7$

We compute

$$\boxed{\mathbb{P}(A_1)} = \mathbb{P}(A_1 \mid A)\,\mathbb{P}(A) + \mathbb{P}(A_1 \mid A^c)\,\mathbb{P}(A^c)$$

$$= .4 \times .3 + .2 \times .7$$

$$\boxed{= 26\%}$$

Example: insurance company (2)

Model: Define

- A_1 = Policy holder has an accident in 1 year
- A = Accident prone

Then

- $S = \{(A_1, A); (A_1^c, A); (A_1, A^c); (A_1^c, A^c)\}$
- Probability: given indirectly by conditioning

Aim:
Compute $\mathbf{P}(A_1)$

Example: insurance company (3)

Given data:

$$\mathbf{P}(A_1 | A) = .4, \qquad \mathbf{P}(A_1 | A^c) = .2, \qquad \mathbf{P}(A) = .3$$

Application of Proposition 3:

$$\mathbf{P}(A_1) = \mathbf{P}(A_1 | A)\,\mathbf{P}(A) + \mathbf{P}(A_1 | A^c)\,\mathbf{P}(A^c)$$

We get

$$\mathbf{P}(A_1) = 0.4 \times 0.3 + 0.2 \times 0.7 = 26\%$$

Example: swine flu (1)

Situation:

We assume that 20% of a pork population has swine flu.
A test made by a lab gives the following results:

- Among 50 tested porks with flu, 2 are not detected
- Among 30 tested porks without flu, 1 is declared sick

Question:

Probability that a pork is healthy while his test is positive?
= probab. to kill the whole population
without proper justification

Events: $F =$ "Flu" $T =$ "Test positive"

Data: $\mathbb{P}(F) = .2$ $\mathbb{P}(F^c) = .8$

$\mathbb{P}(T^c \mid F) = 2/50 = 1/25$

$\mathbb{P}(T \mid F^c) = 1/30$

Test is not very good!

We compute

$\mathbb{P}(F^c \mid T)$

Bayes

$$= \frac{\mathbb{P}(T \mid F^c)\,\mathbb{P}(F^c)}{\mathbb{P}(T \mid F^c)\,\mathbb{P}(F^c) + \mathbb{P}(T \mid F)\,\mathbb{P}(F)}$$

$$= \frac{1/30 \times .8}{1/30 \times .8 + (1 - 1/25) \times .2} = 12\%$$

Example: swine flu (2)

Model: We set F = "Flu", T = "Positive test"
We have

$$\mathbf{P}(F) = \frac{1}{5}, \quad \mathbf{P}(T^c \mid F) = \frac{1}{25}, \quad \mathbf{P}(T \mid F^c) = \frac{1}{30}$$

Aim:
Compute $\mathbf{P}(F^c \mid T)$

Example: swine flu (3)

Application of Proposition 4:

$$\begin{aligned}
\mathbf{P}(F^c \mid T) &= \frac{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c)}{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c) + \mathbf{P}(T \mid F)\,\mathbf{P}(F)} \\
&= \frac{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c)}{\mathbf{P}(T \mid F^c)\,\mathbf{P}(F^c) + [1 - \mathbf{P}(T^c \mid F)]\,\mathbf{P}(F)} \\
&= 0.12
\end{aligned}$$

Conclusion:
12% chance of killing swines without proper justification