Odds

Definition 5.

Let

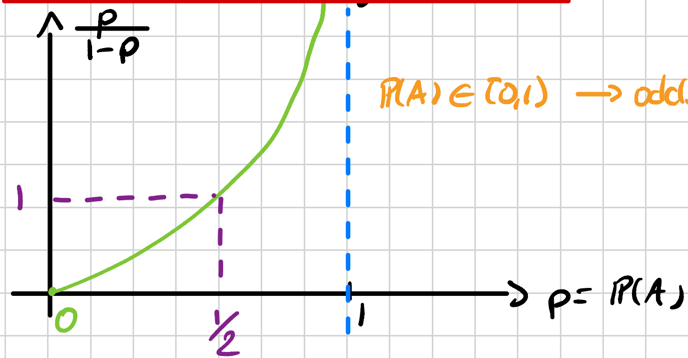
- $\mathbf{P}$ a probability on a sample space S
- A an event

We define the odds of A by

$$\text{odds}(A) = \frac{\mathbf{P}(A)}{\mathbf{P}(A^c)} = \frac{\mathbf{P}(A)}{1 - \mathbf{P}(A)}$$

$odds(A) = \frac{\mathbb{P}(A)}{1-\mathbb{P}(A)}$

Odds as a change of scale



$\mathbb{P}(A) \in [0,1) \longrightarrow odds(A) \in [0,\infty)$

Main use: Betting

Ex: On Wednesday odds(Kamala) = 1.25

Odds and conditioning

Proposition 6.

Situation: We have

- An hypothesis H, true with probability $\mathbf{P}(H)$
- A new evidence E

Formula: The odds of H after evidence E are given by

$$\text{odds}(H|E) = \frac{\mathbf{P}(H|\,E)}{\mathbf{P}(H^c|\,E)} = \frac{\mathbf{P}(H)}{\mathbf{P}(H^c)}\,\frac{\mathbf{P}(E|\,H)}{\mathbf{P}(E|\,H^c)} = \text{odds}(H) \times \text{odd}(E|H)$$

Proof

Inversion of conditioning: We have

$$\mathbf{P}(H|\,E) = \frac{\mathbf{P}(E|\,H)\,\mathbf{P}(H)}{\mathbf{P}(E)}$$

$$\mathbf{P}(H^c|\,E) = \frac{\mathbf{P}(E|\,H^c)\,\mathbf{P}(H^c)}{\mathbf{P}(E)}$$

Conclusion:

$$\frac{\mathbf{P}(H|\,E)}{\mathbf{P}(H^c|\,E)} = \frac{\mathbf{P}(H)}{\mathbf{P}(H^c)}\,\frac{\mathbf{P}(E|\,H)}{\mathbf{P}(E|\,H^c)}$$

Example: coin tossing (1)

Situation:

- Urn contains two type A coins and one type B coin.
- When a type A coin is flipped, it comes up heads with probability $\frac{1}{4}$
- When a type B coin is flipped, it comes up heads with probability $\frac{3}{4}$
- A coin is randomly chosen from the urn and flipped

Question:

Given that the flip landed on heads

$\hookrightarrow$ What is the probability that it was a type A coin?

Data : $\mathbb{P}(H \mid A) = \frac{1}{4}$ $\mathbb{P}(A) = \frac{2}{3}$

$\mathbb{P}(H \mid B) = \frac{3}{4}$ $\mathbb{P}(B) = \frac{1}{3}$

We wish to compute

$\boxed{\mathbb{P}(A \mid H)} \overset{?}{<} \mathbb{P}(A) = \frac{2}{3}$?

$$\overset{\text{Bayes}}{=} \frac{\mathbb{P}(H \mid A)\,\mathbb{P}(A)}{\mathbb{P}(H \mid A)\,\mathbb{P}(A) + \mathbb{P}(H \mid B)\,\mathbb{P}(B)}$$

$$= \frac{\frac{1}{4} \times \frac{2}{3}}{\frac{1}{4} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3}} \boxed{= \frac{2}{5}} < \frac{2}{3}$$

Example: coin tossing (2)

Model: We set

- A = type A coin flipped
- B = type B coin flipped
- H = Head obtained

Data:

$$\mathbf{P}(A) = \frac{2}{3}, \qquad \mathbf{P}(H|\, A) = \frac{1}{4}, \qquad \mathbf{P}(H|\, B) = \frac{3}{4}$$

Aim:

Compute $\mathbf{P}(A|\, H)$

Example: coin tossing (3)

Application of Proposition 6: odd(A) odds(H|A)

$$\text{odds}(A|H) = \frac{\mathbf{P}(A|\, H)}{\mathbf{P}(B|\, H)} = \frac{\mathbf{P}(A)}{\mathbf{P}(B)}\, \frac{\mathbf{P}(H|\, A)}{\mathbf{P}(H|\, B)}$$

Numerical result: We get

$$\frac{\mathbb{P}(A|H)}{1-\mathbb{P}(A|x)} = \frac{\mathbf{P}(A|\, H)}{\mathbf{P}(B|\, H)} = \frac{2/3}{1/3}\,\frac{1/4}{3/4} = \frac{2}{3}$$

Therefore

some algebra

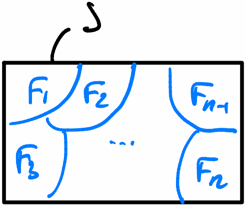
$$\mathbf{P}(A|\, H) = \frac{2}{5}$$

Generalization of Proposition 3

Proposition 7.

Let

- **P** a probability on a sample space S
- $F_1, \ldots, F_n$ partition of S, i.e
  - F_i mutually exclusive
  - $\cup_{i=1}^{n} F_i = S$
- E another event



$n=2$: $F_1, F_2 = F_1^c$ partition

$P(E) = P(E|F_1)\,P(F_1) + P(E|F_2)\,P(F_2)$

Then we have

$$\mathbf{P}(E) = \sum_{i=1}^{n} \mathbf{P}(E|\,F_i)\,\mathbf{P}(F_i)$$

Generalization of Proposition 4

Proposition 8.

Let

- $\mathbf{P}$ a probability on a sample space S
- $F_1, \ldots, F_n$ partition of S, i.e
  - F_i mutually exclusive
  - $\cup_{i=1}^{n} F_i = S$
- E another event

Then we have

$$\mathbf{P}(F_j | E) = \frac{\mathbf{P}(E | F_j)\, \mathbf{P}(F_j)}{\sum_{i=1}^{n} \mathbf{P}(E | F_i)\, \mathbf{P}(F_i)}$$

Example: card game (1)

Situation:

- 3 cards identical in form (say Jack)
- Coloring of the cards on both faces:
  - 1 card RR
  - 1 card BB
  - 1 card RB
- 1 card is randomly selected, with upper side R

Question:

What is the probability that the other side is B?

Example: card game (2)

Model: We define the events

- RR: chosen card is all red
- BB: chosen card is all black
- RB: chosen card is red and black
- R: upturned side of chosen card is red

Aim:
Compute $\mathbf{P}(RB|\, R)$

Example: card game (3)

Application of Proposition 8:

$$\mathbf{P}(RB|\,R) = \frac{\mathbf{P}(R|\,RB)\,\mathbf{P}(RB)}{\mathbf{P}(R|\,RR)\,\mathbf{P}(RR) + \mathbf{P}(R|\,RB)\,\mathbf{P}(RB) + \mathbf{P}(R|\,BB)\,\mathbf{P}(BB)}$$

Numerical values:

$$\mathbf{P}(RB|\,R) = \frac{\frac{1}{2} \times \frac{1}{3}}{1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3}} = \frac{1}{3}$$

Example: disposable flashlights

Situation:

- Bin containing 3 different types of disposable flashlights
- Proba that a type 1 flashlight will give over 100 hours of use is .7
- Corresponding probabilities for types 2 & 3: .4 and .3
- 20% of the flashlights are type 1, 30% are type 2, and 50% are type 3

Questions:

1. What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
2. Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, for $j = 1, 2, 3$?

A = " more than 100k of use " F_j = " Type j chosen "

Data

$\mathbb{P}(A \mid F_1) = .7$ $\mathbb{P}(F_1) = .2$

$\mathbb{P}(A \mid F_2) = .4$ $\mathbb{P}(F_2) = .3$

$\mathbb{P}(A \mid F_3) = .3$ $\mathbb{P}(F_3) = .5$

We wish to compute

$\mathbb{P}(A)$

Bayes I

$$= \mathbb{P}(A \mid F_1)\,\mathbb{P}(F_1) + \mathbb{P}(A \mid F_2)\,\mathbb{P}(F_2) + \mathbb{P}(A \mid F_3)\,\mathbb{P}(F_3)$$

$$= .7 \times .2 + .4 \times .3 + .3 \times .5 = 41\%$$

$\mathbb{P}(A \mid F_1) = .7$ $\quad \mathbb{P}(F_1) = .2$

$\mathbb{P}(A \mid F_2) = .4$ $\quad \mathbb{P}(F_2) = .3$

$\mathbb{P}(A \mid F_3) = .3$ $\quad \mathbb{P}(F_3) = .5$

We wish to compute

$\boxed{\mathbb{P}(F_1 \mid A)} > \mathbb{P}(F_1) = .2\ ?$

Bayes 2

$$= \frac{\mathbb{P}(A \mid F_1)\,\mathbb{P}(F_1)}{\mathbb{P}(A \mid F_1)\,\mathbb{P}(F_1) + \mathbb{P}(A \mid F_2)\,\mathbb{P}(F_2) + \mathbb{P}(A \mid F_3)\,\mathbb{P}(F_3)}$$

computed ⇐ (

$$= \frac{\mathbb{P}(A \mid F_1)\,\mathbb{P}(F_1)}{\mathbb{P}(A)}$$

$$= \frac{.7 \times .2}{.41} \quad \boxed{= .34} > .2$$

Example: disposable flashlights (2)

Model: We define the events

- A: flashlight chosen gives more than 100h of use
- F_j: type j is chosen

Aim 1:
Compute $\mathbf{P}(A)$

Example: disposable flashlights (3)

Application of Proposition 7:

$$\mathbf{P}(A) = \sum_{j=1}^{3} \mathbf{P}(A|\, F_j)\ \mathbf{P}(F_j)$$

Numerical values:

$$\mathbf{P}(A) = 0.7 \times 0.2 + 0.4 \times 0.3 + 0.3 \times 0.5 = .41$$

Example: disposable flashlights (4)

Aim 2:
Compute $\mathbf{P}(F_1|\, A)$

Application of Proposition 8:

$$\mathbf{P}\left(F_1|\, A\right) = \frac{\mathbf{P}\left(A|\, F_1\right)\mathbf{P}\left(F_1\right)}{\mathbf{P}(A)}$$

Numerical value:

$$\mathbf{P}\left(F_1|\, A\right) = \frac{0.7 \times 0.2}{0.41} = \frac{14}{41} \simeq 41\%$$

Outline

Definition of independence

Definition 9.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events

Then E and F are independent if

$$\mathbf{P}(E\,F) = \mathbf{P}(E)\,\mathbf{P}(F)$$

Notation:

E and F independent denoted by $E \perp\!\!\!\perp F$

Def $E \perp\!\!\!\perp F$ if $\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)$

Rmk 1 If $E \perp\!\!\!\perp F$, we also have

$$\boxed{\mathbb{P}(E \mid F) = \mathbb{P}(E)}$$

Rmk 2

$E \perp\!\!\!\perp F$ is very different from $E \cap F = \emptyset$