Definition of independence

Definition 9.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events

Then E and F are independent if

$$\mathbf{P}(E\,F) = \mathbf{P}(E)\,\mathbf{P}(F)$$

Notation:

E and F independent denoted by $E \perp\!\!\!\perp F$

Def $E \perp\!\!\!\perp F$ if $\mathbb{P}(E \cap F) = \mathbb{P}(E)\,\mathbb{P}(F)$

Rmk 1 If $E \perp\!\!\!\perp F$, we also have

$$\boxed{\mathbb{P}(E \mid F) = \mathbb{P}(E)}$$

Rmk 2

$E \perp\!\!\!\perp F$ is very different from $E \cap F = \emptyset$

Rmk 1 If $E \perp\!\!\!\perp F$, we also have

$$\boxed{\mathbb{P}(E \mid F) = \mathbb{P}(E)}$$

The information in F does not change the chances of E

Intuition of $\perp\!\!\!\perp$

Proof If $E \perp\!\!\!\perp F$. Then

$$\mathbb{P}(E \mid F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}$$

$$\overset{\perp\!\!\!\perp}{=} \frac{\mathbb{P}(E)\,\cancel{\mathbb{P}(F)}}{\cancel{\mathbb{P}(F)}} = \mathbb{P}(E)$$

This never happens

Rmk 2

$E \perp\!\!\!\perp F$ is very different from $E \cap F = \emptyset$

Assume $\boxed{E \perp\!\!\!\perp F \text{ and } E \cap F = \emptyset}$. Then

(i) $\mathbb{P}(E \cap F) \overset{\perp\!\!\!\perp}{=} \mathbb{P}(E)\,\mathbb{P}(F)$

(ii) $\mathbb{P}(E \cap F) = \mathbb{P}(\emptyset) = 0$

Thus $\mathbb{P}(E)\,\mathbb{P}(F) = 0$

$\Rightarrow$ either $E = \emptyset$ or $F = \emptyset$

Some remarks

Interpretation: If $E \perp\!\!\!\perp F$, then

$$\mathbf{P}\left(E \middle| F\right) = \mathbf{P}(E),$$

that is the knowledge of F does not affect $\mathbf{P}(E)$

Warning: Independent $\neq$ mutually exclusive!
Specifically

$$\begin{aligned} A, B \text{ mutually exclusive} \quad &\Rightarrow \quad \mathbf{P}(A\,B) = 0 \\ A, B \text{ independent} \quad &\Rightarrow \quad \mathbf{P}(A\,B) = \mathbf{P}(A)\,\mathbf{P}(B) \end{aligned}$$

Therefore A et B both independent and mutually exclusive
$\hookrightarrow$ we have either $\mathbf{P}(A) = 0$ or $\mathbf{P}(B) = 0$

Example: dice tossing (1)

Experiment: We throw two dice

Sample space:

- $S = \{1, \ldots, 6\}^2$
- $\mathbf{P}(\{(s_1, s_2)\}) = \frac{1}{36}$ for all $(s_1, s_2) \in S$

Events: We consider

$$A = \text{"1}^{\text{st}} \text{ outcome is 1"}, \qquad B = \text{"2}^{\text{nd}} \text{ outcome is 4"}$$

Question:
Do we have $A \perp\!\!\!\perp B$?

Intuition ?

Yes

$\boxed{A}$ = "Outcome 1st roll is 1"

$= \{1\} \times \{1, \dots, 6\}$

$= \{(1,1); \dots; (1,6)\}$

$\mathbb{P}(A) = \frac{|A|}{36} = \frac{6}{36} = \frac{1}{6}$

$\boxed{B}$ = "Outcome 2nd roll is 4"

$= \{1, \dots, 6\} \times \{4\}$

$\mathbb{P}(B) = \frac{|B|}{36} = \frac{1}{6}$

$\boxed{A \cap B} = \{(1, 4)\}$

$\mathbb{P}(A \cap B) = \frac{1}{36}$

Conclusion

$$\frac{1}{36} = \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B) = \frac{1}{6} \times \frac{1}{6}$$

$$\Rightarrow \quad \boxed{A \perp\!\!\!\perp B}$$

Example: dice tossing (2)

Description of A and B:

$$A = \{1\} \times \{1, \ldots, 6\}, \quad \text{and} \quad B = \{1, \ldots, 6\} \times \{4\}.$$

Probabilities for A and B: We have

$$\mathbf{P}(A) = \frac{|A|}{36} = \frac{1}{6}, \qquad \mathbf{P}(B) = \frac{|B|}{36} = \frac{1}{6}$$

Description of $A\,B$: We have $A\,B = \{(1,4)\}$. Thus

$$\mathbf{P}(A\,B) = \frac{1}{36} = \mathbf{P}(A)\,\mathbf{P}(B)$$

Conclusion: A and B are independent

Example: tossing n coins (1)

$S = \{h, t\}^n \qquad \mathbb{P}(\{s\}) = \frac{1}{2^n}$

Experiment:

Tossing a coin n times

Events: We consider

Few h

A = "At most one Head"

B = "At least one Head and one Tail"

more h

Question:

Are there values of n such that $A \perp\!\!\!\perp B$?

Intuition $\longrightarrow$ No

$A = (A \cap B) \cup (A \cap B^c)$

$\boxed{A} =$ "At most 1 h"

$= \{ (t, \dots, t); (h, t, \dots, t); (t, h, t, \dots, t); \dots (t, \dots, t, h) \}$

$|A| = n+1$ $\qquad \mathbb{P}(A) = \frac{n+1}{2^n}$

$\boxed{B} =$ "At least 1 h and 1 t"

$B^c = \{ (t, \dots, t); (h, \dots, h) \}$

$\mathbb{P}(B) = 1 - \mathbb{P}(B^c) = 1 - \frac{2}{2^n} = 1 - \frac{1}{2^{n-1}}$

$\boxed{A \cap B} = A \setminus (A \cap B^c) = A \setminus \{ (t, \dots, t) \}$

$\mathbb{P}(A \cap B) = \frac{n}{2^n}$

We have $\mathbb{P}(A) = \frac{n+1}{2^n}$ $\mathbb{P}(B) = 1 - \frac{1}{2^{n-1}}$

$\mathbb{P}(A \cap B) = \frac{n}{2^n}$

Indep. $A \perp\!\!\!\perp B \iff \mathbb{P}(A \cap B) = \mathbb{P}(A)\,\mathbb{P}(B)$

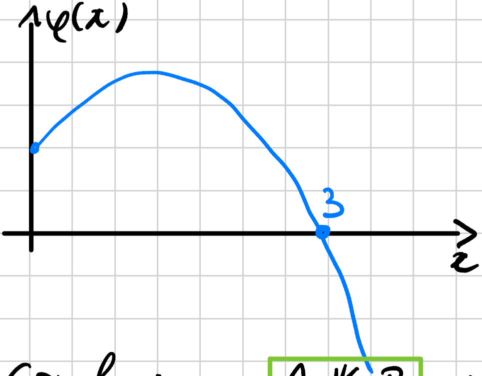
$$\iff \frac{n}{2^n} = \frac{n+1}{2^n}\left(1 - \frac{1}{2^{n-1}}\right)$$

(some algebra)

$$\iff \underbrace{n - 2^{n-1} + 1}_{\varphi(n)} = 0$$

Equation : $\varphi(n)=0$ with

$$\varphi(x) = x - 2^{x-1} + 1 = 0$$



Conclusion $\boxed{A \not\Rightarrow B}$ unless $n = 3$

↳ our intuition is correct most of the time

Example: tossing n coins (2)

Model: We take

- $S = \{h, t\}^n$
- $\mathbf{P}(\{s\}) = \frac{1}{2^n}$ for all $s \in S$

Description of A and B:

$$
\begin{aligned}
A &= \{(t, \ldots, t), (h, t, \ldots, t), (t, h, t, \ldots, t), (t, \ldots, t, h)\} \\
B &= \{(h, \ldots, h), (t, \ldots, t)\}^c
\end{aligned}
$$

Example: tossing n coins (3)

Computing probabilities for A and B: We have

$$\begin{aligned} \mathbf{P}(A) &= \frac{|A|}{2^n} = \frac{n+1}{2^n} \\ \mathbf{P}(B) &= 1 - \mathbf{P}(B^c) = 1 - \frac{1}{2^{n-1}} \end{aligned}$$

Description of $A\,B$ and

$$A\,B = A \backslash \{(t, \ldots, t)\} \quad \Rightarrow \quad \mathbf{P}(A\,B) = \frac{n}{2^n}$$

Example: tossing n coins (4)

Checking independence: We have $A \perp\!\!\!\perp B$ iff

$$\frac{n+1}{2^n}\left(1-\frac{1}{2^{n-1}}\right)=\frac{n}{2^n} \quad\Longleftrightarrow\quad n-2^{n-1}+1=0$$

Conclusion: One can check that

$$x \mapsto x-2^{x-1}+1$$

vanishes for $x=3$ only on $\mathbb{R}_+$. Thus

We have $A \perp\!\!\!\perp B$ iff $n=3$

Independence and complements

Proposition 10.

Let

- $\mathbf{P}$ a probability on a sample space S
- E, F two events
- We assume that $E \perp\!\!\!\perp F$

Then

$$E \perp\!\!\!\perp F^c, \qquad E^c \perp\!\!\!\perp F, \qquad E^c \perp\!\!\!\perp F^c$$