Rmk We can have A, B, C s.t.

$A \perp\!\!\!\perp B$, $A \perp\!\!\!\perp C$, $B \perp\!\!\!\perp C$

But

A, B, C are NOT $\perp\!\!\!\perp$

Example

$S = \{1, \dots, 6\}^2 \quad \mathbb{P}(\{s\}) = \frac{1}{36}$

A = " even outcome, 1st roll "
B = " odd outcome, 2nd roll "
C = " same parity for the 2 rolls "

$A = \{2, 4, 6\} \times \{1, \dots, 6\} \Rightarrow \mathbb{P}(A) = \frac{|A|}{36} = \frac{18}{36} = \frac{1}{2}$

$B = \{1, \dots, 6\} \times \{1, 3, 5\} \Rightarrow \mathbb{P}(B) = \frac{|B|}{36} = \frac{1}{2}$

$C = \{2, 4, 6\}^2 \cup \{1, 3, 5\}^2 \Rightarrow \mathbb{P}(C) = \frac{|C|}{36} = \frac{9+9}{36} = \frac{1}{2}$

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$$A \cap B = \{2,4,6\} \times \{1,3,5\}$$

$$\Rightarrow \mathbb{P}(A \cap B) = \frac{|A \cap B|}{36} = \frac{9}{36} = \frac{1}{4} = \mathbb{P}(A)\mathbb{P}(B)$$

$\Uparrow$

$\boxed{A \perp\!\!\!\perp B}$

$$A \cap C = \{2,4,6\}^2$$

$$\Rightarrow \mathbb{P}(A \cap C) = \frac{|A \cap C|}{36} = \frac{9}{36} = \frac{1}{4} = \mathbb{P}(A)\mathbb{P}(C)$$

$\Uparrow$

$\boxed{A \perp\!\!\!\perp C}$

Check We also have $\boxed{B \perp\!\!\!\perp C}$

$A = \{2, 4, 6\} \times \{1, \dots, 6\} \Rightarrow \mathbb{P}(A) = \frac{|A|}{36} = \frac{18}{36} = \frac{1}{2}$

$B = \{1, \dots, 6\} \times \{1, 3, 5\} \Rightarrow \mathbb{P}(B) = \frac{|B|}{36} = \frac{1}{2}$

$C = \{2, 4, 6\}^2 \cup \{1, 3, 5\}^2 \Rightarrow \mathbb{P}(C) = \frac{|C|}{36} = \frac{9+9}{36} = \frac{1}{2}$

However

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\emptyset) = 0$$

$$\neq \mathbb{P}(A)\,\mathbb{P}(B)\,\mathbb{P}(C) = \frac{1}{8}$$

Thus $\boxed{A, B, C \text{ are not } \perp\!\!\!\perp}$

Counterexample: independence of 3 events (1)

Warning:
In certain situations we have A, B, C pairwise independent, however

$$\mathbf{P}(A \cap B \cap C) \neq \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C)$$

Example: tossing two dice

- $S = \{1, \ldots, 6\}^2$
- $\mathbf{P}(\{(s_1, s_2)\}) = \frac{1}{36}$ for all $(s_1, s_2) \in S$

Events: Define

$$\begin{aligned} A &= \text{"even number for the 1}^{\text{st}}\text{ outcome"} \\ B &= \text{"odd number for the 2}^{\text{nd}}\text{ outcome"} \\ C &= \text{"same parity for the two outcomes"} \end{aligned}$$

Counterexample: independence of 3 events (2)

Description of A, B, C:

$$\begin{aligned} A &= \{2,4,6\} \times \{1,\ldots,6\} \\ B &= \{1,\ldots,6\} \times \{1,3,5\} \\ C &= (\{2,4,6\} \times \{2,4,6\}) \cup (\{1,3,5\} \times \{1,3,5\}) \end{aligned}$$

Pairwise independence: we find

$$A \perp\!\!\!\perp B, \; A \perp\!\!\!\perp C \text{ and } B \perp\!\!\!\perp C$$

Independence of the 3 events: We have $A \cap B \cap C = \emptyset$. Thus

$$0 = \mathbf{P}(A \cap B \cap C) \neq \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C) = \frac{1}{8}$$

Independence of 3 events

Definition 11.

Let

- $\mathbf{P}$ a probability on a sample space S
- 3 events A_1, A_2, A_3

We say that A_1, A_2, A_3 are independent if

$$\begin{aligned}\mathbf{P}(A_1A_2) &= \mathbf{P}(A_1)\,\mathbf{P}(A_2), \quad \mathbf{P}(A_1A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_3) \\ \mathbf{P}(A_2A_3) &= \mathbf{P}(A_2)\,\mathbf{P}(A_3)\end{aligned}$$

and

$$\mathbf{P}(A_1A_2A_3) = \mathbf{P}(A_1)\,\mathbf{P}(A_2)\,\mathbf{P}(A_3)$$

Independence of n events

Definition 12.

Let

- $\mathbf{P}$ a probability on a sample space S
- n events $A_1, A_2, \ldots, A_n$

We say that $A_1, A_2, \ldots, A_n$ are independent if
for all $2 \le r \le n$ and $j_1 < \cdots < j_r$ we have

$$\mathbf{P}(A_{j_1} A_{j_2} \cdots A_{j_r}) = \mathbf{P}(A_{j_1})\, \mathbf{P}(A_{j_2}) \cdots \mathbf{P}(A_{j_r})$$

Independence of an ∞ number of events

Definition 13.

Let

- $\mathbf{P}$ a probability on a sample space S
- A sequence of events $\{A_i;\, i \geq 1\}$

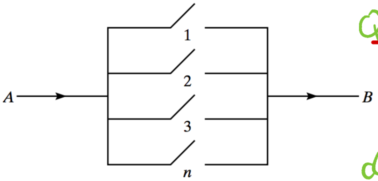
We say that the A_i's are independent if
for all $2 \leq r < \infty$ and $j_1 < \cdots < j_r$ we have

$$\mathbf{P}(A_{j_1} A_{j_2} \cdots A_{j_r}) = \mathbf{P}(A_{j_1})\, \mathbf{P}(A_{j_2}) \cdots \mathbf{P}(A_{j_r})$$

Example: parallel system (1)

Situation:

- Parallel system with n components
- All components are independent Hyp
- Probability that i-th component works: p_i



Question: how is possible that the system does not work?

if none of them works

Question:

Probability that the system functions

Events $A =$ " system functions "
$A_i =$ " i-th component works "

Hyp A_i's are $\perp\!\!\!\perp$ (see Def 12)
$\mathbb{P}(A_i) = p_i$

Then

$\perp\!\!\!\perp$ if A_i's $\perp\!\!\!\perp$

$$\boxed{\mathbb{P}(A)} = 1 - \mathbb{P}(A^c) = 1 - \mathbb{P}\left(\bigcap_{i=1}^{n} A_i^c\right)$$

$$= 1 - \prod_{i=1}^{n} \mathbb{P}(A_i^c)$$

$$\boxed{= 1 - \prod_{i=1}^{n} (1 - p_i)}$$

Example: parallel system (2)

Model: We take

- $S = \{0,1\}^n$
- Probability $\mathbf{P}$ on S defined by

$$\mathbf{P}(\{(s_1,\ldots,s_n)\}) = \prod_{i=1}^{n} p_i^{s_i}(1-p_i)^{1-s_i}$$

Events:

$$A = \text{"System functions"}\ , \quad A_i = \text{"}i\text{-th component functions"}$$

Facts about A_i's:

The events A_i are independent and $\mathbf{P}(A_i) = p_i$

Example: parallel system (3)

Computations for $\mathbf{P}(A^c)$:

$$\begin{aligned} \mathbf{P}(A^c) &= \mathbf{P}\left(\cap_{i=1}^n A_i^c\right) \\ &= \prod_{i=1}^n \mathbf{P}\left(A_i^c\right) \\ &= \prod_{i=1}^n (1 - p_i) \end{aligned}$$

Conclusion:

$$\mathbf{P}(A) = 1 - \prod_{i=1}^n (1 - p_i)$$

Example: rolling dice (1)

Experiment:

- Roll a pair of dice
- Outcome: sum of faces

Event: We define

- $E =$ "5 appears before 7" $=$ " $S=5$ appears before $S=7$ "

Question:
Compute $\mathbf{P}(E)$

Events $E =$ " $S = 5$ before $S = 7$ "

random variable ← $S_i =$ "sum, i-th roll"

Then we have Hyp: all $\perp\!\!\!\perp$

$$E = \bigcup_{n=1}^{\infty} \left((S_1 \neq 5,7) \cap (S_2 \neq 5,7) \cdots \cap (S_{n-1} \neq 5,7) \cap (S_n = 5) \right)$$

disjoint U

$$\boxed{\mathbb{P}(E)} = \sum_{n=1}^{\infty} \mathbb{P}((S_1 \neq 5,7) \cap \cdots \cap (S_{n-1} \neq 5,7) \cap (S_n = 5))$$

$$\overset{\perp\!\!\!\perp}{=} \sum_{n=1}^{\infty} \mathbb{P}(S_1 \neq 5,7) \times \cdots \times \mathbb{P}(S_{n-1} \neq 5,7) \times \mathbb{P}(S_n = 5)$$

check

$$= \sum_{n=1}^{\infty} \left(\frac{13}{18}\right)^{n-1} \frac{1}{9} \boxed{= \frac{2}{5}}$$

(geometric series)

2nd method: Condition on the first roll

→ Markov chain technique