Example: rolling dice (1)

Experiment:

- Roll a pair of dice
- Outcome: sum of faces

Event: We define

- $E =$ "5 appears before 7" = " $S=5$ appears before $S=7$ "

Question:
Compute $\mathbf{P}(E)$

2nd method: Condition on the first roll

→ Markov chain technique

Events: E = " $S=5$ before $S=7$ "

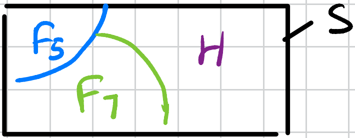
We condition on

F_5 = " $S=5$ on 1st roll "

F_7 = " $S=7$ on 1st roll "

H = " $S \neq 5, 7$ on 1st roll "

Rmk F_5, F_7, H partition of S



F_5 : " $S=5$ on 1st roll "
F_7 : " $S=7$ on 1st roll "
H : " $S\neq 5,7$ on 1st roll "

We can use Bayes 1 :

$$\mathbb{P}(E) = \Big(\overbrace{\mathbb{P}(E \mid F_5)}^{=1}\, \mathbb{P}(F_5)\Big)\ \tfrac{1}{9}$$
$$+ \overbrace{\mathbb{P}(E \mid F_7)}^{=0}\, \mathbb{P}(F_7)$$
$$+ \Big(\mathbb{P}(E \mid H)\, \mathbb{P}(H)\Big)\ \tfrac{13}{18}$$

Claim : $\mathbb{P}(E \mid H) \underset{\text{by } \perp\!\!\!\perp}{\overset{\Uparrow}{=}} \mathbb{P}(E)$

$E \cap H$ = " $S \neq 5,7$ on 1st roll " $\Big) \perp\!\!\!\perp$

$\cap$ " $S=5$ before $S=7$ on 2nd roll onward "

We have:

$$\mathbb{P}(E) = \overbrace{\mathbb{P}(E \mid F_5)}^{=1}\,\mathbb{P}(F_5)\Big)\,\tfrac{1}{9}$$
$$+ \overbrace{\mathbb{P}(E \mid F_7)}^{=0}\,\mathbb{P}(F_7)$$
$$+ \underbrace{\mathbb{P}(E \mid H)}_{=\,\mathbb{P}(E)}\,\mathbb{P}(H)\Big)\,\tfrac{13}{18}$$

Thus

$$\mathbb{P}(E) = \frac{1}{9} + \frac{13}{18}\,\mathbb{P}(E)$$

$\Rightarrow$ $\mathbb{P}(E)$ solves $\quad x = \frac{1}{9} + \frac{13}{18}x$

$$\Rightarrow \quad \boxed{\mathbb{P}(E) = \frac{2}{5}}$$

Advantage: no geometric series

Random variables

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Probability - MA 416

Mostly taken from *A first course in probability*
by S. Ross

Outline

1. Random variables
2. Discrete random variables
3. Expected value
4. Expectation of a function of a random variable
5. Variance
6. The Bernoulli and binomial random variables
7. The Poisson random variable
8. Other discrete random variables
9. Expected value of sums of random variables
10. Properties of the cumulative distribution function

Outline

So far: sample space S, event, $\mathbb{P}$

Instead of S, we might be interested in an "observable" of S
numerical quantity

Random variable X: function

$$X : S \longrightarrow \mathbb{R}$$

In calculus:

$$f : \mathbb{R} \longrightarrow \mathbb{R}$$

Example: Toss 3 coins $S = \{h, t\}^3$

Define $X = \#$ heads

s	$X(s)$	s	$X(s)$
(h,h,h)	3	(t,h,h)	2
(h,h,t)	2	(t,h,t)	1
(h,t,h)	2	(t,t,h)	1
(h,t,t)	1	(t,t,t)	0

Introduction

Experiment: tossing 3 coins

Model:

$$S = \{h, t\}^3, \ \mathbf{P}(\{s\}) = \frac{1}{8} \text{ for all } s \in S$$

Result of the experiment: we are interested in the quantity

$$X(s) = \text{"\# Heads obtained when } s \text{ is realized"}$$

Introduction (2)

Table for the outcomes:

s	$X(s)$	s	$X(s)$
(t, t, t)	0	(h, t, t)	1
(t, t, h)	1	(h, t, h)	2
(t, h, t)	1	(h, h, t)	2
(t, h, h)	2	(h, h, h)	3

Introduction (3)

Information about X:

X is considered as an application, i.e.

$$X : S \to \{0, 1, 2, 3\}.$$

Aim: "forget" about S

Then we wish to understand sets like

$$X^{-1}(\{2\}) = \{(t, h, h), (h, t, h), (h, h, t)\}$$

or quantities like

$$= \frac{|\{(t,h,h), (h,t,h), (h,h,t)\}|}{8}$$

$$\mathbb{P}(X = 2) = \mathbf{P}\left(X^{-1}(\{2\})\right) = \frac{3}{8}.$$

This will be formalized in this chapter

Example: time of first success (1)

Experiment:

- Coin having probability p of coming up heads
- Independent trials: flipping the coin
- Stopping rule: either H occurs or n flips made

Random variable:

$$X = \# \text{ of times the coin is flipped}$$

State space:

$$X \in \{1, \ldots, n\}$$

Computing $\mathbb{P}(X=j)$, $j=1,\ldots,n-1$

$$\mathbb{P}(X=j)=\mathbb{P}(\{(\overbrace{t,t,\ldots,t}^{j-1},h)\})$$

$$\overset{\perp\!\!\!\perp}{=}(1-p)^{j-1}\,p$$

Computing $\mathbb{P}(X=n)$

$$\mathbb{P}(X=n)=\mathbb{P}(\{(\overbrace{t,\ldots,t}^{n-1},h);(\overbrace{t,\ldots,t}^{n})\})$$

$$\overset{\perp\!\!\!\perp}{=}(1-p)^{n-1}\,p+(1-p)^{n}$$

$$=(1-p)^{n-1}\,(p+1-p)$$

$$=(1-p)^{n-1}$$

Example: time of first success (2)

Probabilities for $j < n$:

$$\mathbf{P}(X = j) = \mathbf{P}\left(\{(t, \ldots, t, h)\}\right) = (1-p)^{j-1} p$$

Probability for $j = n$:

$$\mathbf{P}(X = n) = \mathbf{P}\left(\{(t, \ldots, t, h); (t, \ldots, t, t)\}\right) = (1-p)^{n-1}$$