Example: time of first success (1)

Experiment:

- Coin having probability p of coming up heads
- Independent trials: flipping the coin
- Stopping rule: either H occurs or n flips made

Random variable:

$$X = \# \text{ of times the coin is flipped}$$

State space:

$$X \in \{1, \ldots, n\}$$

Example: time of first success (2)

Probabilities for $j < n$:

$$\mathbf{P}(X = j) = \mathbf{P}\left(\{(t, \ldots, t, h)\}\right) = (1 - p)^{j-1} p$$

Probability for $j = n$:

$$\mathbf{P}(X = n) = \mathbf{P}\left(\{(t, \ldots, t, h); (t, \ldots, t, t)\}\right) = (1 - p)^{n-1}$$

Example: time of first success (3)

Checking the sum of probabilities:

$$\begin{aligned}
\mathbf{P}\left(\bigcup_{j=1}^{n}\{X=j\}\right) &= \sum_{j=1}^{n}\mathbf{P}\left(\{X=j\}\right) \\
&= p\sum_{j=1}^{n-1}(1-p)^{j-1}+(1-p)^{n} \\
&= 1
\end{aligned}$$

Security check: we should always have

$$\sum_{j=1}^{n} \mathbb{P}(X=j) = 1$$

Here

$$\boxed{\sum_{j=1}^{n} \mathbb{P}(X=j)} = \sum_{j=1}^{n-1} p(1-p)^{j-1} + (1-p)^{n-1}$$

$\vdots$ some algebra

$$\boxed{= 1}$$

Next aim. Find a proper description of a random variable

We will see two possibilities

(i) Cumulative distribution function (cdf)

(ii) Probability mass function (pmf)

Cumulative distribution function

Here X is described by a function $F: \mathbb{R} \longrightarrow [0,1]$

Definition 1.

Let

- $\mathbf{P}$ a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable, with $\mathcal{E} \subset \mathbb{R}$

For $x \in \mathbb{R}$ we define

$$F(x) = \mathbf{P}(X \leq x)$$

Then the function F is called cumulative distribution function or distribution function

Rmk The Pmf is only defined for discrete random variables

Discrete r.v $X: S \longrightarrow E$ where E is a countable set

$\exists$ a one-to-one map $E \longrightarrow \mathbb{N}$

Examples: $\{1, \dots, n\}^d$, $\mathbb{N}^d$, even numbers, $\mathbb{Z}^d$, $\mathbb{Q}$

Sets which are not countable:

$\mathbb{R}^d$, $\mathbb{C}^d$

Outline

1. Random variables
2. Discrete random variables
3. Expected value
4. Expectation of a function of a random variable
5. Variance
6. The Bernoulli and binomial random variables
7. The Poisson random variable
8. Other discrete random variables
9. Expected value of sums of random variables
10. Properties of the cumulative distribution function

General definition

Definition 2.

Let

- **P** a probability on a sample space S
- $X : S \to \mathcal{E}$ a random variable

Hypothesis: $\mathcal{E}$ is countable, i.e

$$\mathcal{E} = \{x_i; i \geq 1\}$$

Then we say that X is a discrete random variable

Probability mass function

Here X is described by a sequence $\{p(x_i); i \geq 1\}$

Definition 3.

Let

- $\mathbf{P}$ a probability on a sample space S
- $\mathcal{E} = \{x_i; i \geq 1\}$ countable state space
- $X : S \to \mathcal{E}$ discrete random variable

For $i \geq 1$ we set

$$p(x_i) = \mathbf{P}(X = x_i)$$

Then the probability mass function of X is the family

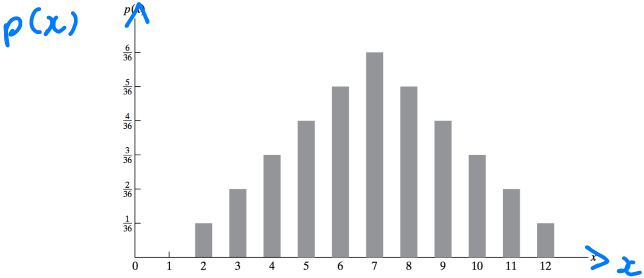
$$\{p(x_i);\ i \geq 1\}$$

Remarks

Sum of the pmf: If p is the pmf of X, then

$$\sum_{i \geq 1} p(x_i) = 1$$

Graph of a pmf: Bar graphs are often used.
Below an example for $X =$ sum of two dice



Example of pmf computation (1)

Definition of the pmf: Let X be a r.v with pmf given by

$\mathcal{E} = \{0, 1, 2, \ldots\}$

$$p(i) = c\,\frac{\lambda^i}{i!}, \qquad i \geq 0,$$

where $c > 0$ is a normalizing constant

Question: Compute

1. $\mathbf{P}(X = 0)$
2. $\mathbf{P}(X > 2)$

R.V X. Given by $\mathbb{P}(X=i) = c\,\dfrac{\lambda^{i}}{i!} \equiv p(i)$, $\quad i=0,1,\dots$

(1) Computation of c. We have to check

$$\sum_{i=0}^{\infty} p(i) = 1$$

Here

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} c\,\frac{\lambda^{i}}{i!} = c\overbrace{\sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!}}^{=e^{\lambda}} = c e^{\lambda}$$

Thus

$$\sum_{i=1}^{\infty} p(i) = 1 \iff c e^{\lambda} = 1 \iff \boxed{c = e^{-\lambda}}$$

Summary $\mathbb{P}(X=i) = e^{-\lambda}\frac{\lambda^i}{i!} \equiv p(i)$

(2) Compute $\mathbb{P}(X=0)$

$$\boxed{p(0)} = \mathbb{P}(X=0) = e^{-\lambda}\frac{\lambda^0}{0!} \boxed{= e^{-\lambda}}$$

(3) Compute $\mathbb{P}(X>2)$

$$\mathbb{P}(X>2) = \sum_{i=3}^{\infty}\mathbb{P}(X=i) = \sum_{i=3}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!}$$

Simpler: $\boxed{\mathbb{P}(X>2)} = 1 - \mathbb{P}(X\leq 2)$

$$= 1 - (p(0) + p(1) + p(2))$$

$$\boxed{= 1 - \left(e^{-\lambda} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!}\right)}$$

We have seen

$$\mathbb{P}(X>2) = 1 - \left(e^{-\lambda} + e^{-\lambda}\frac{\lambda^1}{1!} + e^{-\lambda}\frac{\lambda^2}{2!}\right)$$

Thus

$$\mathbb{P}(X>2) = 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

Example of pmf computation (2)

Computing c: We must have

$$c \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = 1$$

Thus

$$c = e^{-\lambda}$$

Computing $\mathbf{P}(X = 0)$: We have

$$\mathbf{P}(X = 0) = e^{-\lambda} \frac{\lambda^0}{0!} = e^{-\lambda}$$

Example of pmf computation (3)

Computing $\mathbf{P}(X > 2)$: We have

$$\mathbf{P}(X > 2) = 1 - \mathbf{P}(X \le 2)$$

Thus

$$\mathbf{P}(X > 2) = 1 - e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2}\right)$$

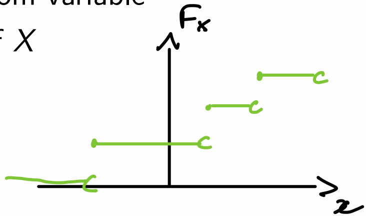
Cdf for discrete random variables

Proposition 4.

Def : $F_X(x) = \mathbb{P}(X \leq x)$

Let

- $\mathbf{P}$ a probability on a sample space S
- $\mathcal{E} = \{x_i; i \geq 1\}$ countable state space, with $\mathcal{E} \subset \mathbb{R}$
- $X : S \to \mathcal{E}$ discrete random variable
- F_X cdf of X and p pmf of X



Then

1. F can be expressed as

$$F(a) = \sum_{i \geq 1;\, x_i \leq a} p(x_i)$$

2. F is a step function