Outline

Expected value for discrete random variables

Definition 5.

Let

- $\mathbf{P}$ a probability on a sample space S
- $\mathcal{E} = \{x_i; i \geq 1\}$ countable state space, with $\mathcal{E} \subset \mathbb{R}$
- $X : S \to \mathcal{E}$ discrete random variable
- p pmf of X

weighted avg. of x_i

Then we define

$$\mathbf{E}[X] = \sum_{i \geq 1} x_i \, \mathbf{P}(X = x_i) = \underline{\sum_{i \geq 1} x_i \, p(x_i)}$$

weights

Justification of the definition

Experiment:

- Run independent copies of the random variable X
- For i-th copy, the measurement is z_i

Result (to be proved much later):

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} z_i = \mathbf{E}[X]$$

Example: dice rolling (1)

Definition of the random variable: we consider

$$X = \text{outcome when we roll a fair dice}$$

Example: dice rolling (2)

$\mathbb{E}(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + \cdots + 6 \times \frac{1}{6}$

$= \frac{1}{6}(1+2+\cdots+6) = \frac{1}{\cancel{6}} \times \frac{(1+6) \times \cancel{6}}{2} = \frac{7}{2}$

Recall: we consider

$$X = \text{outcome when we roll a fair dice}$$

sample space

Pmf: We have $\mathcal{E} = \{1, \ldots, 6\}$ and

$$p(1) = \cdots = p(6) = \frac{1}{6}$$

Expected value: We get

$$\mathbf{E}[X] = \sum_{i=1}^{6} i\, p(i) = \frac{1}{6} \sum_{i=1}^{6} i = \frac{7}{2}$$

Example: indicator of an event (1)

Definition of the random variable:

Let A event with $\mathbf{P}(A) = p$ and set

1_A

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

p.m.f.

$Pr(1_A = 1) = P(A) = p$

$Pr(1_A = 0) = 1 - P(A) = 1 - p$

$\mathbb{E}(1_A) = 1 \times p + 0 \times (1-p)$

$= p$

$= P(A)$

Example: indicator of an event (2)

Recall:

Let A event with $\mathbf{P}(A) = p$ and set

$$\mathbf{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{if } A^c \text{ occurs} \end{cases}$$

Pmf:

$$p(0) = 1 - p, \qquad p(1) = p$$

Expected value:

$$\mathbf{E}[\mathbf{1}_A] = p$$

Outline

First attempt of a definition

Problem: Let

- X discrete random variable
- $Y = g(X)$ for a function g

How can we compute $\mathbf{E}[g(X)]$?

First strategy:

- $Y = g(X)$ is a discrete random variable
- Determine the pmf p_Y of Y
- Compute $\mathbf{E}[Y]$ according to Definition 5

First attempt: example (1)

① p.m.f for $g(X) = Y$

② $\mathbb{E}(Y)$

$$X = \begin{cases} -1 & \text{w.p. } 0.2 \\ 0 & \text{w.p. } 0.5 \\ 1 & \text{w.p. } 0.3 \end{cases}$$

Definition of a random variable X:

Let $X : S \to \{-1, 0, 1\}$ with

$$\mathbf{P}(X = -1) = .2, \quad \mathbf{P}(X = 0) = .5, \quad \mathbf{P}(X = 1) = .3$$

We wish to compute $\mathbf{E}[X^2]$

$Y = X^2$, p.m.f. for Y

$$Y = \begin{cases} 1 & \text{w.p. } 0.2 \\ 0 & \text{w.p. } 0.5 \\ 1 & \text{w.p. } 0.3 \end{cases}$$

$$Y = \begin{cases} 1 & \text{w.p. } 0.5 \\ 0 & \text{w.p. } 0.5 \end{cases} \Longleftarrow$$

$1 \times 0.5 + 0 \times 0.5$
$= 0.5$

$\mathbb{E}(Y) = 0.5 = Pr(Y=1)$

First attempt: example (2)

Definition of a random variable Y: Set $Y = X^2$.
Then $Y \in \{0, 1\}$ and

$$\begin{aligned} \mathbf{P}(Y = 0) &= \mathbf{P}(X = 0) = .5 \\ \mathbf{P}(Y = 1) &= \mathbf{P}(X = -1) + \mathbf{P}(X = 1) = .5 \end{aligned}$$

First attempt: example (3)

Recall: For $Y = X^2$ we have

$$\mathbf{P}(Y = 0) = .5, \qquad \mathbf{P}(Y = 1) = .5$$

Expected value:

$$\mathbf{E}\left[X^2\right] = \mathbf{E}[Y] = .5$$

Definition of $\mathbf{E}[g(X)]$

Proposition 6.

Let

- X discrete random variable
- p pmf of X
- g real valued function

$$\mathbb{E}(X) = \sum_{i \geq 1} x_i \, p(x_i)$$

Then

$$\mathbf{E}\left[g(X)\right] = \sum_{i \geq 1} g(x_i)\, p(x_i) \qquad (1)$$

Let $g(X) = Y$

$$E(Y) = \sum_{j \geq 1} y_j \Pr(Y = y_j)$$

$$X = \begin{cases} -1 \\ 0 \\ 1 \end{cases}$$

$$\Pr(Y = y_j) = \boxed{\sum_{i : g(X_i) = y_j} \Pr(X = x_i)}$$

$$Y = X^2$$

$$\Pr(Y=1) = \Pr(X=-1) + \Pr(X=1)$$

$$E(Y) = \sum_{j \geq 1} y_j \sum_{i : g(X_i) = y_j} \Pr(X = x_i)$$

$$= \sum_{j \geq 1} \sum_{i : g(X_i) = y_j} y_j \Pr(X = x_i)$$

$$= \underbrace{\sum_{j \geq 1} \sum_{i : g(X_i) = y_j}} \underbrace{g(X_i) \Pr(X = x_i)}_{\text{no } j \text{ index}} = \sum_{i \geq 1} g(X_i) \overbrace{\Pr(X = X_i)}^{p(X_i)}$$

Proof

Values of Y: We set $Y = g(X)$ and

$$\{y_j;\, j \ge 1\} = \text{ values of } g(x_i) \text{ for } i \ge 1$$

Expression for the rhs of (1): gather according to y_j

$$\begin{aligned}
\sum_{i\ge1} g(x_i)\, p(x_i) &= \sum_{j\ge1} \sum_{i;\, g(x_i)=y_j} y_j\, p(x_i) \\
&= \sum_{j\ge1} y_j \sum_{i;\, g(x_i)=y_j} p(x_i) \\
&= \sum_{j\ge1} y_j\, \mathbf{P}\left(g(X) = y_j\right) \\
&= \sum_{j\ge1} y_j\, \mathbf{P}\left(Y = y_j\right) \\
&= \mathbf{E}\left[g(X)\right]
\end{aligned}$$

Previous example reloaded

$E(X^2) = (-1)^2 \times 0.2 + 0^2 \times 0.5 + 1^2 \times 0.3 = 0.5$

Definition of a random variable X:

Let $X : S \to \{-1, 0, 1\}$ with

$$\mathbf{P}(X = -1) = .2, \quad \mathbf{P}(X = 0) = .5, \quad \mathbf{P}(X = 1) = .3$$

We wish to compute $\mathbf{E}[X^2]$

Application of (1):

$$\mathbf{E}\left[X^2\right] = \sum_{i=-1,0,1} i^2 p(x_i) = .5$$

Example: seasonal product (1)

If $X > s$, then all the stocks will be sold making a profit of $\boxed{b \cdot s}$

If $X \le s$, then X units sold, $(s-X)$ unsold

profit bX

loss $(s-X)\ell$

net profit: $\boxed{bX - (s-X)\ell}$

Situation:

- Product sold seasonally
- Profit b for each unit sold
- Loss ℓ for each unit left unsold
- Product has to be stocked in advance
  $\hookrightarrow$ s units stocked

Random variable:

- $X = \#$ units of product ordered by customers
- Pmf p for X

Question:

Find optimal s in order to maximize expected profits

Example: seasonal product (2)

$\mathbb{E}(Y_{s+1}) - \mathbb{E}(Y_s) > 0$

< 0

Some random variables: We set

$\mathcal{D}\ \mathcal{E} = \{0, 1, 2, \ldots\}$

$$\begin{aligned} X &= \text{\# units ordered, with pmf } p \\ Y_s &= \text{profit when } s \text{ units stocked} \end{aligned}$$

Expression for Y_s:

indicator

$$Y_s = (b\,X - (s - X)\,\ell)\,\mathbf{1}_{(X \le s)} + s\,b\,\mathbf{1}_{(X > s)} = g(X)$$

Expression for $\mathbf{E}[Y_s]$:

$$sb\left[\underbrace{\sum_{i=0}^{\infty} p(i)}_{=1} - \sum_{i=0}^{s} p(i)\right]$$

$$\mathbf{E}[Y_s] = \sum_{i=0}^{s} (b\,i - (s - i)\,\ell)\,p(i) + \sum_{i=s+1}^{\infty} s\,b\,p(i)$$

$\rightarrow \sum_{i=0}^{s}$

Simplify

$\mathbb{E}(Y_{s+1})$

$$E(Y_s) = \sum_{i=0}^{s} \left(bi - (s-i)l\right) p(i) + sb\left[1 - \sum_{i=0}^{s} p(i)\right]$$

$$= sb + \sum_{i=0}^{s} \left[bi - (s-i)l - sb\right] p(i)$$

$$bi - sl + il - sb$$

$$= b(i-s) + l(i-s) = (i-s)(b+l)$$

$$= sb + \sum_{i=0}^{s} (i-s)(b+l)\, p(i)$$

Example: seasonal product (3)

Simplification for $\mathbf{E}[Y_s]$: We get

$$\mathbf{E}\left[Y_s\right] = s\,b + (b+\ell)\sum_{i=0}^{s}(i-s)\,p(i)$$

Growth of $s \mapsto \mathbf{E}[Y_s]$: We have

$$\mathbf{E}\left[Y_{s+1}\right] - \mathbf{E}\left[Y_s\right] = b - (b+\ell)\sum_{i=0}^{s} p(i)$$