Let X r.v. with pmf p. Then

$$\boxed{\begin{aligned} \mathbb{E}[X] &= \sum_i x_i \, p(x_i) \\ \mathbb{E}[g(X)] &= \sum_i g(x_i) \, p(x_i) \end{aligned}}$$

Example If $X : S \to \{1,2\}$ with pmf $p(1) = \frac{1}{2}$, $p(2) = \frac{1}{2}$. Then

$$\mathbb{E}[X] = \sum_{i=1}^{2} i \, p(i) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = \frac{3}{2}$$

$$\mathbb{E}[X^2] = \sum_{i=1}^{2} i^2 \, p(i) = 1^2 \times \frac{1}{2} + 2^2 \times \frac{1}{2} = \frac{5}{2}$$

Example: seasonal product (1)

Situation:

- Product sold seasonally
- Profit b for each unit sold
- Loss ℓ for each unit left unsold
- Product has to be stocked in advance
  $\hookrightarrow$ s units stocked

Random variable:

- $X = \#$ units of product ordered
- Pmf p for X

Question:

Find optimal s in order to maximize profits

Example: seasonal product (2)

Some random variables: We set

$$
\begin{aligned}
X &= \#\text{ units ordered, with pmf } p \\
Y_s &= \text{profit when } s \text{ units stocked}
\end{aligned}
$$

Expression for Y_s:

$$Y_s = \overbrace{(b\,X - (s-X)\,\ell)\,\mathbf{1}_{(X\le s)} + s\,b\,\mathbf{1}_{(X>s)}}^{g(x)}$$

Expression for $\mathbf{E}[Y_s]$:

$$= E[g(x)] = \sum_{i=0}^{\infty} g(i)\,p(i)$$

$$\mathbf{E}[Y_s] = \sum_{i=0}^{s} (b\,i - (s-i)\,\ell)\,p(i) + \sum_{i=s+1}^{\infty} s\,b\,p(i)$$

Example: seasonal product (3)

Simplification for $\mathbf{E}[Y_s]$: We get

$$\mathbf{E}[Y_s] = s\,b + (b+\ell)\sum_{i=0}^{s}(i-s)\,p(i)$$

Growth of $s \mapsto \mathbf{E}[Y_s]$: We have

$$\mathbf{E}[Y_{s+1}] - \mathbf{E}[Y_s] = b - (b+\ell)\sum_{i=0}^{s} p(i)$$

Example: seasonal product (4)

Growth of $s \mapsto \mathbf{E}[Y_s]$ (Ctd): We obtain

$$\mathbf{E}[Y_{s+1}] - \mathbf{E}[Y_s] > 0 \quad \Longleftrightarrow \quad \sum_{i=0}^{s} p(i) < \frac{b}{b+\ell} \tag{2}$$

Optimization:

- The lhs of (2) is $\nearrow$
- The rhs of (2) is constant
- Thus there exists a s^* such that

$$\mathbf{E}[Y_0] < \cdots < \mathbf{E}[Y_{s^*-1}] < \mathbf{E}[Y_{s^*}] > \mathbf{E}[Y_{s^*+1}] > \cdots$$

Conclusion: s^* leads to maximal expected profit

Expectation and linear transformations

Proposition 7.

Let

- X discrete random variable
- p pmf of X
- $a, b \in \mathbb{R}$ constants

Then

$$\mathbf{E}[aX + b] = a\,\mathbf{E}[X] + b$$

Proof

Application of relation (1):

$$\begin{aligned}
\mathbf{E}\left[aX+b\right] &= \sum_{i\geq 1}(a\,x_i+b)\;p(x_i) \\
&= a\sum_{i\geq 1}x_i\,p(x_i)+b\sum_{i\geq 1}p(x_i) \\
&= a\,\mathbf{E}\left[X\right]+b
\end{aligned}$$

Outline

Rmk If we want a short description of a r.v X, we can use

(i) $E[X] \rightarrow$ average value of X

(ii) $(Var(X))^{\frac{1}{2}} = (V(X))^{\frac{1}{2}}$

$\hookrightarrow$ average fluctuation of X

Definition of variance

Definition 8.

Let

- X discrete random variable
- p pmf of X
- $\mu = \mathbf{E}[X]$

Then we define $\mathbf{Var}(X)$ by $= E[\,g(X)\,]$ with $g(x) = (x-\mu)^2$

$$\mathbf{Var}(X) = \mathbf{E}\left[(X-\mu)^2\right]$$

Interpretation

Expected value: For a r.v X, $\mathbf{E}[X]$ represents the mean value of X.

Variance: For a r.v X, $\mathbf{Var}(X)$ represents the dispersion of X wrt its mean value.

A greater $\mathbf{Var}(X)$ means

- The system represented by X has a lot of randomness
- This system is unpredictable

Standard deviation: For physical reasons, it is better to introduce

$$\sigma_X := \sqrt{\mathbf{Var}(X)}.$$

Example: Consider 2 soccer players.
Observe the # goals scored by those 2 players during the 5 last games

Palmer → reliable
Anthony → random

Palmer	1	1	1	1	1
Anthony	5	0	0	0	0

⚠ This is stat, not probab.

Q: How to choose between P and A according to our data?

Palmer	1	1	1	1	1
Anthony	5	0	0	0	0

1st step : Compute averages

$$\bar{x}_P = \frac{1}{5} \sum_{i=1}^{5} x_i = \frac{1}{5}(1 + \cdots + 1) = 1 \text{ goal/game}$$

$$\bar{x}_A = \frac{1}{5}(5 + 0 + \cdots + 0) = 1 \text{ goal/game}$$

$$\boxed{\text{No way to distinguish between P and A by just looking at } \bar{x}\text{ !}}$$

Palmer	1	1	1	1	1
Anthony	5	0	0	0	0

2nd step . Compute fluctuations

$$\mathrm{Var}_P = \frac{1}{5}\sum_{i=1}^{5}(x_i-1)^2 = \frac{1}{5}\left((1-1)^2+\cdots(1-1)^2\right)$$

$$= 0 \rightarrow \text{no fluctuation}$$

$$\mathrm{Var}_A = \frac{1}{5}\left\{(5-1)^2+4\times(0-1)^2\right\} = \frac{1}{5}\times 20$$

$$= 4 \ (\text{goals}/\text{game})^2$$

large fluctuations!

$$\sigma_A = (\mathrm{Var}_A)^{\frac{1}{2}} = 2 \text{ goals}/\text{game}$$

Conclusion

P very reliable

A very random

Interpretation (2)

Illustration (from descriptive stats): We wish to compare the performances of 2 soccer players on their last 5 games

Griezmann	5	0	0	0	0
Messi	1	1	1	1	1

Recall: for a set of data $\{x_i;\, i \le n\}$, we have
Empirical mean: $\bar{x}_n = \frac{1}{n}\sum_{i=1}^n x_i$
Empirical variance: $s_n^2 = \frac{1}{n}\sum_{i=1}^n (x_i - \bar{x}_n)^2$
Standard deviation: $s_n = \sqrt{s_n^2}$

On our data set: $\bar{x}_G = \bar{x}_M = 1$ goal/game
$\hookrightarrow$ Same goal average
However, $s_G = 2$ goals/game while $s_M = 0$ goals/game
$\hookrightarrow$ M more reliable (less random) than G

Alternative expression for the variance

Proposition 9.

$Var(X) = E[(X-\mu)^2]$

Let

- X discrete random variable
- p pmf of X
- $\mu = \mathbf{E}[X]$

Then $\mathbf{Var}(X)$ can be written as

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - \mu^2 = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

Example $X =$ Outcome when rolling 1 dice

pmf: $p(1) = \frac{1}{6}$, ..., $p(6) = \frac{1}{6}$

$$\boxed{E[X]} = \sum_{i=1}^{6} i\, p(i) = \frac{1}{6} \sum_{i=1}^{6} i = \frac{1}{6} \times \frac{6 \times 7}{2}$$

$$= \frac{7}{2} \boxed{= 3.5}$$

$$E[X^2] = \sum_{i=1}^{6} i^2 p(i) = \frac{1}{6}(1 + 4 + 9 + 16 + 25 + 36)$$

$$= \frac{91}{6}$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\boxed{\sigma_X} = \left(\frac{35}{12}\right)^{\frac{1}{2}} \boxed{\simeq 1.71}$$

large fluctuations wrt the scale of X

Example: rolling a dice

Random variable:

- $X =$ outcome when one rolls 1 dice

Variance computation: We find

$$\mathbf{E}[X] = \frac{7}{2}, \qquad \mathbf{E}[X^2] = \frac{91}{6}$$

Therefore

$$\mathbf{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

Standard deviation:

$$\sigma_X = \sqrt{\frac{35}{12}} \simeq 1.71$$