Monday

- $V(X) = \mathbb{E}[(X-\mu)^2]$, $\mu = \mathbb{E}[X]$
- Interpretation : "amount of randomness"
- $V(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

Variance and linear transformations

Recall : $\mathbb{E}[aX+b] = a\,\mathbb{E}[X]+b$

Proposition 10.

Let

- X discrete random variable
- p pmf of X
- $a, b \in \mathbb{R}$ constants

b affects the mean,
not the fluctuations

Then

$$\mathbf{Var}(aX+b) = a^2\,\mathbf{Var}(X)$$

Outline

Bernoulli random variable (1)

Notation:

distributed like Bernoulli parameter p

$$X \sim \mathcal{B}(p) \text{ with } p \in (0,1)$$

State space:

$$\{0,1\}$$

Pmf:

$$p(0) = \mathbf{P}(X=0) = 1-p, \quad \mathbf{P}(X=1) = p = p(1)$$

Expected value and variance:

$$\mathbf{E}[X] = p, \qquad \mathbf{Var}(X) = p(1-p)$$

⚠ You have to know how to compute this type of quantity

Bernoulli random variable (2)

Use 1, success in a binary game:

- Example 1: coin tossing
  - $X = 1$ if H, $X = 0$ if T
  - We get $X \sim \mathcal{B}(1/2)$

$p = \frac{1}{2}$ if coin is fair
$p = \mathbb{P}(\text{Head})$

- Example 2: dice rolling
  - $X = 1$ if outcome $= 3$, $X = 0$ otherwise
  - We get $X \sim \mathcal{B}(1/6)$

$p = \mathbb{P}(\{3\}) = \frac{1}{6}$

Use 2, answer yes/no in a poll

- $X = 1$ if a person feels optimistic about the future
- $X = 0$ otherwise
- We get $X \sim \mathcal{B}(p)$, with unknown p

Question: how to estimate p based on polling $n = 1000$ persons ?

Let $X \sim \mathcal{B}(p)$. Then

(i) $$\boxed{\mathbb{E}[X]} = \sum_{i=0}^{1} i \, p(i) = 0 \times p(0) + 1 \times p(1)$$

$$= \overbrace{0 \times (1-p)}^{0} + \overbrace{1 \times p}^{p} \quad \boxed{= p}$$

(ii) $$\mathbb{E}[X^2] = \sum_{i=0}^{1} i^2 \, p(i)$$

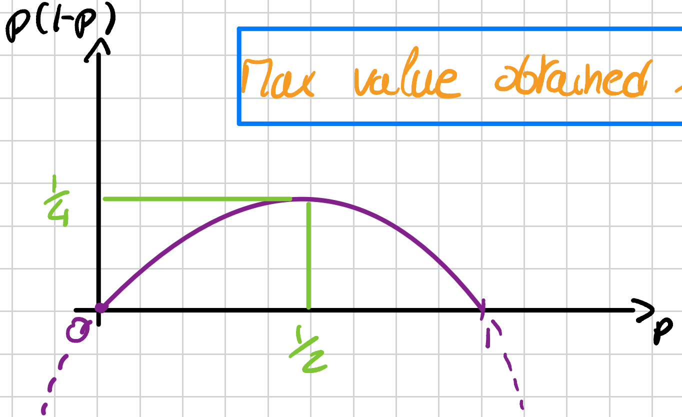
$$= \overbrace{0^2 \times (1-p)}^{0} + \overbrace{1^2 \times p}^{p} = p$$

(iii) $$\boxed{\mathrm{Var}(X)} = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= p - p^2 \quad \boxed{= p(1-p)}$$

<u>Question</u>: If we look $p \mapsto B(p)$, when do we have the "max amount of randomness" ?

i.e when is $V(X) = p(1-p)$ max ?



$$\boxed{\text{Max value obtained for } p = \tfrac{1}{2}}$$

Jacob Bernoulli

Flip a coin $n = 1000$ times
Then # Heads $\simeq$ 500

Some facts about Bernoulli:

- Lifespan: 1654-1705, in Switzerland
- Discovers constant e
- Establishes divergence of $\sum \frac{1}{n}$
- Contributions in diff. eq
- First law of large numbers
- Bernoulli:
  family of 8 prominent mathematicians
- Fierce math fights between brothers

Binomial random variable (1)

Notation:

$$X \sim \text{Bin}(n, p), \text{ for } n \geq 1,\ p \in (0, 1)$$

State space:

$$\{0, 1, \ldots, n\}$$

Pmf:

$$\mathbf{P}(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \quad 0 \leq k \leq n$$

Expected value and variance:

$$\mathbf{E}[X] = np, \qquad \mathbf{Var}(X) = np(1 - p)$$

Binomial random variable (2)

$n = 9$

Use 1, Number of successes in a Bernoulli trial:

- Example: Roll a dice 9 times.
- $X = \#$ of 3 obtained
- We get $X \sim \text{Bin}(9, 1/6)$
- $\mathbf{P}(X = 2) = 0.28 = \binom{9}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^7$

$p = \frac{1}{6} = \mathbb{P}(\{3\})$

28% chance to obtain a 3 twice when we roll the dice 9 times

$E[X] = 9 \times \frac{1}{6} = \frac{3}{2}$ $\quad V(X) = 9 \times \frac{1}{6} \times \frac{5}{6} = \frac{3}{2} \times \frac{5}{6} = \frac{5}{4}$

Use 2: Counting a feature in a repeated trial:

- Example: stock of 1000 pants with 10% defects $\quad p = .1$
- Draw 15 $= n$ times a pant at random
- $X = \#$ of pants with a defect
- We get $X \sim \text{Bin}(15, 1/10)$

$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

Let $X \sim Bin(n, p)$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Security check: $\sum_{k=0}^{n} p(k) = 1$?

Here

$$\boxed{\sum_{k=0}^{n} p(k)} = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial formula

$$= [p + (1-p)]^n$$

$$= 1^n \boxed{= 1} \longrightarrow \text{OK!}$$