Outline

Basic principle of counting

Theorem 1.

Suppose 2 experiments to be performed and

- For Experiment 1, we have m possible outcomes
- For each outcome of Experiment 1
  $\hookrightarrow$ We have n outcomes for Experiment 2

Then

Total number of possible outcomes is $m \times n$

Proof

Sketch of the proof: Set

$$(i, j) \equiv \text{Outcome } i \text{ for Experiment 1 \& Outcome } j \text{ for Experiment 2}$$

Then enumerate possibilities

Application of basic principle of counting

Example: Small community with

- 10 women
- Each woman has 3 chidren

We have to pick one pair as mother & child of the year

Question:

How many possibilities?

Mother and child example

Experiment 1: pick a mother

\# outcomes is $m = 10$

Experiment 2: pick a child

\# outcomes is $n = 3$

Total # of outcomes: $m \times n = 10 \times 3$

$= 30$

Generalized principle of counting

Theorem 2.

Suppose r experiments to be performed and

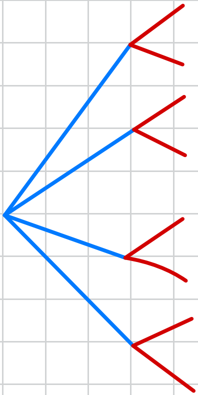
- For Experiment 1, we have n_1 possible outcomes
- For each outcome of Experiment i
  $\hookrightarrow$ We have n_{i+1} outcomes for Experiment $i+1$

Then total number of possible outcomes is

$$\prod_{i=1}^{r} n_i = n_1 \times n_2 \times \cdots \times n_r$$

Illustration with tree

Take $r = 2$, $n_1 = 4$, $n_2 = 2$



$\#$ of outcomes $= 8$

$= 4 \times 2$

Application of basic principle of counting

Example 1: Find # possible 7 place license plates if

- First 3 places are letters
- Final 4 places are numbers

Answer: 175,760,000

Example 2: Find # possible 7 place license plates if

- First 3 places are letters
- Final 4 places are numbers
- No repetition among letters or numbers

Answer: 78,624,000

License plate example

- Experiment 1 to 3: pick a letter

  → $n_1 = 26$ $n_2 = 26$ $n_3 = 26$

- Experiment 4 to 7: pick a number

  → $n_4 = 10$ $n_5 = 10$ $n_6 = 10$ $n_7 = 10$

Total # of outcomes

$$= \prod_{i=1}^{7} n_i = 26^3 \times 10^4 = 175{,}760{,}000$$

License plate example - no repetition

- **Experiment 1 to 3:** pick a letter

  → $n_1 = 26$ $n_2 = 25$ $n_3 = 24$

- **Experiment 4 to 7:** pick a number

  → $n_4 = 10$ $n_5 = 9$ $n_6 = 8$ $n_7 = 7$

Total # of outcomes

$$= \prod_{i=1}^{7} n_i = 26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7$$

$$= 78{,}624{,}000$$

Outline

Permutations

Definition:

A permutation of n objects is an ordered sequence of those n objects.

Property:

Two permutations only differ according to the order of the objects

Counting:

Let P_n be the number of permutations for n objects. Then

$$P_n = n! = n \times (n-1) \cdots \times 2 = \prod_{j=1}^{n} j$$

Basic example of permutation (1)

Example: 3 balls, Red, Black, Green

Can we enumerate the permutations?

Permutations of R B G $\quad n = 3$

R B G	B R G	G R B
R G B	B G R	G B R

# of permutations :

$$\boxed{P_3 = 3!} = 3 \times 2 = 6 \text{ possibilities}$$

OK

Basic example of permutation (2)

Example: 3 balls, Red, Black, Green

Permutations: RBG, RGB, BRG, BGR, GBN, GBR
$\hookrightarrow$ 6 possibilities

Formula: $P_3 = 3! = 6$

Proof for the counting number P_n

Sketch of the proof:
Direct application of Theorem 2

Example of permutation (1)

Problem:

Count possible arrangements of letters in PEPPER

Permutations of PEPPER $n=6$

Version 1: ≠ letters, $P_1 E_1 P_2 P_3 E_2 R_1$

Then # permutations $= \boxed{P_6 = 6! = 720}$

Version 2: P's and E's cannot be distinguished

Then # permutations

$$= \frac{6!}{\underbrace{3!}_{\#P's}\ \underbrace{2!}_{\#E's}} = \boxed{60}$$

Example of permutation (2)

Solution 1:

Consider all letters as distinct objects

$$P_1\ E_1\ P_2\ P_3\ E_2\ R$$

Then

$$P_6 = 6! = 720 \text{ possibilities}$$

Example of permutation (3)

Solution 2:

Do not distinguish P's and E's.

Then

$$\frac{P_6}{P_3\, P_2} = \frac{6!}{3!\, 2!} = 60 \text{ possibilities}$$

Outline

Combinations

Definition:

A combination of p objects among n objects is non ordered subset of p objects.

Property:

Two combinations only differ according to nature of their objects

Counting:

The number of combinations of p objects among n objects is

$$\binom{n}{p} = \frac{n!}{p!\,(n-p)!}$$

Basic example of combination

Example: 4 balls, Red, Black, Green, Purple
We pick 2 balls in this group

Can we enumerate the combinations?

Example of combination: R B G P → $n = 4$
Pick $p = 2$ of them

Version 1: we do care about the order

R B	B R	G R	P R
R G	B G	G B	P B
R P	B P	G P	P G

12 outcomes $= 12$

$= \frac{4!}{2!}$

This called arrangement, # arr. $= \frac{n!}{(n-p)!}$

Version 2 : order does not matter → combination

combinations $= \frac{n!}{p!\,(n-p)!} = \frac{4!}{2!\,2!} = 6$ → OK