Let $X \sim \mathrm{Bin}(n,p)$. Then

$$E[X] = \sum_{k=0}^{n} k \; p(k) = \sum_{k=1}^{n} k \; p(k)$$

$$= \sum_{k=1}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \qquad p\,p^{k-1} \quad (n-1)-(k-1)$$

$$= \sum_{k=1}^{n} k \frac{\overset{n(n-1)!}{n!}}{\underbrace{k!}_{(k-1)!}\underbrace{(n-k)!}_{((n-1)-(k-1))!}} p^k (1-p)^{\overset{(n-1)-(k-1)}{n-k}}$$

$$= \sum_{k=1}^{n} \frac{n(n-1)!}{(k-1)!\,((n-1)-(k-1))!} \; p \; p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$= n p \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$$\boxed{E[X]}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} p^{k-1} (1-p)^{n-1-(k-1)}$$

$$= np \sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{(n-1)-(k-1)}$$

$l = k-1$

$$= np \sum_{l=0}^{n-1} \binom{n-1}{l} p^{l} (1-p)^{(n-1)-l}$$

Binomial

$$= np \, [p + (1-p)]^{n-1}$$

$$\boxed{= np}$$

Binomial random variable (3)

$n = 6$, $p = 0.5$

$E[X] = 6 \times .5 = 3$

n small

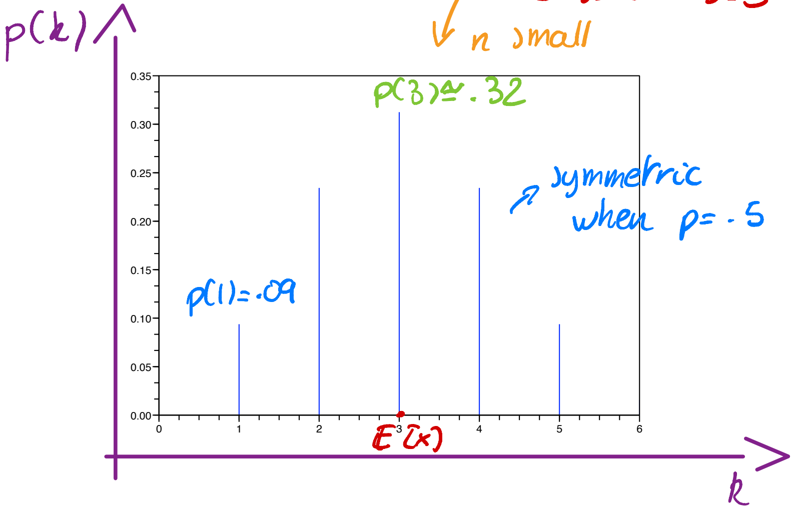

Figure: Pmf for Bin(6; 0.5). x-axis: k. y-axis: $\mathbf{P}(X = k)$

Binomial random variable (4)

$n = 30$, $p = 0.5$

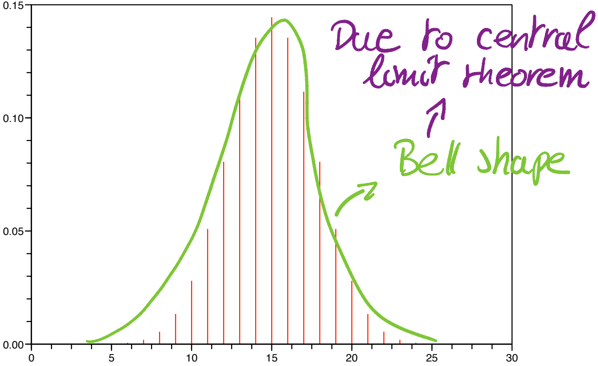

Figure: Pmf for $\text{Bin}(30; 0.5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Example: wheel of fortune (1)

Game:

- Player bets on $1, \ldots, 6$ (say 1)
- 3 dice rolled
- If 1 does not appear, loose \$1
- If 1 appear i times, win \$i

Question:
Find average win

> 0
$= 0$?
< 0

Model for the game

Roll dice 3 times
↳ # 1's

(a) Consider

$X = \#\ 1\text{'s}$. Then $X \sim \mathrm{Bin}(3, \frac{1}{6})$

(b) Notation: For an event A, set

$$\mathbb{1}_A = \begin{cases} 1 & \text{if } A \text{ occurs} \\ 0 & \text{otherwise} \end{cases}$$

Then $\boxed{\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)}$

(c) Rule: If $X = 0$, then $W = -1$

If $X = 1, 2, 3$, then $W = X$

Thus $W = X \mathbb{1}_{(X \neq 0)} - \mathbb{1}_{(X=0)}$

$= X - \mathbb{1}_{(X=0)}$

(d) We wish to compute $X \sim Bin(3, \frac{1}{6})$

$\boxed{\mathbb{E}[W]} = \mathbb{E}[X] - \mathbb{E}[\mathbb{1}_{(X=0)}]$

$= \mathbb{E}[X] - \mathbb{P}(X = 0)$

$= 3 \times \frac{1}{6} - \overbrace{\binom{3}{0}}^{1} \overbrace{\left(\frac{1}{6}\right)^0}^{1} \left(\frac{5}{6}\right)^3 = \frac{1}{2} - \left(\frac{5}{6}\right)^3 \boxed{= -0.079}$

Example: wheel of fortune (2)

Binomial random variable:

- Let $X = \#$ times 1 appears
- Then $X \sim \text{Bin}(3, \frac{1}{6})$

Expression for the win: Set $W =$ win. Then

- $W = \varphi(X)$ with
  $\hookrightarrow$ $\varphi(0) = -1$ and $\varphi(i) = i$ for $i = 1, 2, 3$
- Other expression:

$$W = X - \mathbf{1}_{(X=0)}$$

Example: wheel of fortune (3)

Average win:

$$\begin{aligned} \mathbf{E}[W] &= \mathbf{E}[X] - \mathbf{P}(X=0) \\ &= \frac{1}{2} - \left(\frac{5}{6}\right)^3 \\ &= -\frac{17}{216} \end{aligned}$$

Conclusion: The average win is

$$\mathbf{E}[W] \simeq -\$0.079$$

Pmf variations for a binomial r.v

Proposition 11.

Let

- $X \sim \mathrm{Bin}(n, p)$
- $q =$ Pmf of X
- $k^* = \lfloor (n+1)p \rfloor$

Then we have

- $k \mapsto q(k)$ is $\nearrow$ if $k < k^*$
- $k \mapsto q(k)$ is $\searrow$ if $k > k^*$
- Maximum of q attained for $k = k^*$

Proof

Pmf computation: We have

$$\frac{q(k)}{q(k-1)} = \frac{\mathbf{P}(X=k)}{\mathbf{P}(X=k-1)} = \frac{(n-k+1)p}{k(1-p)}$$

Pmf growth: We get

$$\mathbf{P}(X=k) \geq \mathbf{P}(X=k-1) \quad \Longleftrightarrow \quad k \leq (n+1)p$$