Outline

Poisson random variable (1)

Notation:

$$\mathcal{P}(\lambda) \text{ for } \lambda \cancel{\in} \mathbb{R}_+ \quad \lambda > 0$$

State space:

$$E = \mathbb{N} \cup \{0\}$$ (1st example of infinite state space)

Pmf:

$$\mathbf{P}(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k \geq 0$$

Expected value and variance:

$$\mathbf{E}[X] = \lambda, \qquad \mathbf{Var}(X) = \lambda$$

Poisson random variable (2)

Use (examples):

- # customers getting into a shop from 2pm to 5pm
- # buses stopping at a bus stop in a period of 35mn
- # jobs reaching a server from 12am to 6am

→ Queuing theory M/M/1

Empirical rule:

If $n \to \infty$, $p \to 0$ and $np \to \lambda$, we approximate $\text{Bin}(n, p)$ by $\mathcal{P}(\lambda)$. This is usually applied for

$$p \leq 0.1 \quad \text{and} \quad np \leq 5$$

" $\text{Bin}(n, \frac{\lambda}{n}) \xrightarrow{n \to \infty} \mathcal{P}(\lambda)$ "

$\mathbb{E}[X]$ for $X \sim P(\lambda)$

$$\boxed{\mathbb{E}[X]} = \sum_{k=0}^{\infty} k \; p(k)$$

$$= \sum_{k=1}^{\infty} k \; e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^k}{(k-1)!} \qquad \lambda^{k-1} \times \lambda$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} \qquad \text{cv: } \ell = k-1$$

$$= \lambda e^{-\lambda} \underbrace{\sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}}_{e^{\lambda}} \qquad \boxed{= \lambda}$$

Trick : We compute $E[X(X-1)]$

$$\boxed{E[X(X-1)]} = \sum_{k=0}^{\infty} k(k-1)\, e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^k}{(k-2)!} \qquad \lambda^{k-2} \times \lambda^2$$

$$= \lambda^2 e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} \qquad \text{cv: } \ell = k-2$$

$$= \lambda^2 e^{-\lambda} \sum_{\ell=0}^{\infty} \frac{\lambda^\ell}{\ell!}$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} \qquad \boxed{= \lambda^2}$$

Summary. We have proved

$$\mathbb{E}[X] = \lambda \qquad \mathbb{E}[X(X-1)] = \lambda^2$$

Next aim: get $\mathbb{E}[X^2]$

$$\lambda^2 = \mathbb{E}[X(X-1)] = \mathbb{E}[X^2 - X]$$

$\mathbb{E}$ linear

$$= \mathbb{E}[X^2] - \mathbb{E}[X]$$

$$\Rightarrow \boxed{\mathbb{E}[X^2]} = \lambda^2 + \mathbb{E}[X]$$

$$\boxed{= \lambda^2 + \lambda}$$

New summary

$\mathbb{E}[X]$, $\mathbb{E}[X^2] = \lambda^2 + \lambda$

$Var(X)$ for $X \sim \mathcal{P}(\lambda)$

$$Var(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\Rightarrow \boxed{Var(X) = \lambda}$$

Poisson random variable (3)

$\mathcal{P}(2)$

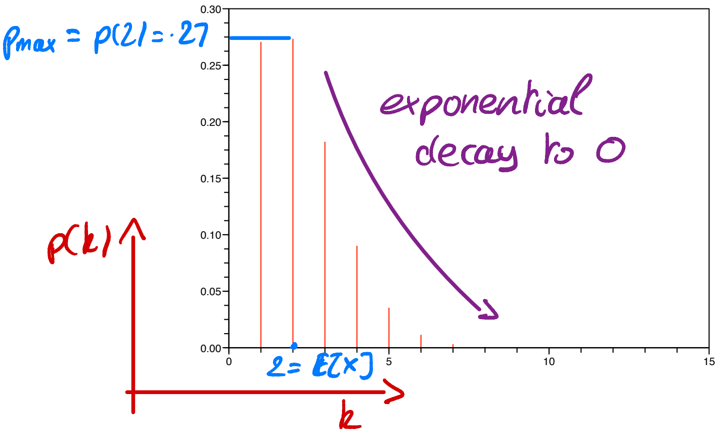


Figure: Pmf of $\mathcal{P}(2)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Poisson random variable (4)

Poisson = Fish

$\mathcal{P}(5)$

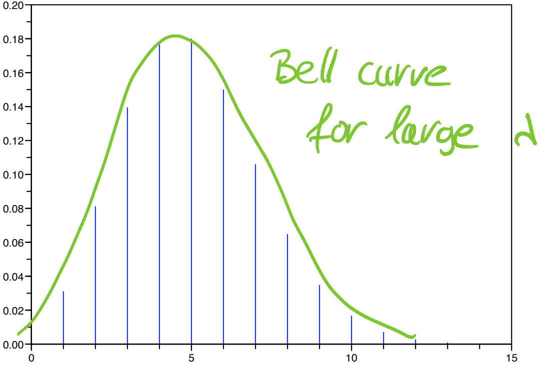


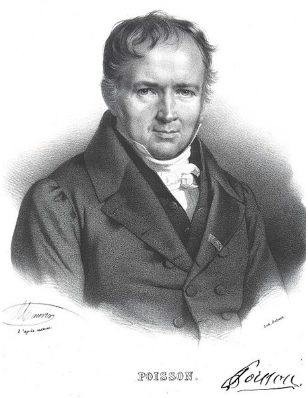
Figure: Pmf of $\mathcal{P}(5)$. x-axis: k. y-axis: $\mathbf{P}(X = k)$

Siméon Poisson

Some facts about Poisson:

- Lifespan: 1781-1840, in $\simeq$ Paris
- Engineer, Physicist and Mathematician
- Breakthroughs in electromagnetism
- Contributions in partial diff. eq
  celestial mechanics, Fourier series
- Marginal contributions in probability

$$\text{Bin}(n, \tfrac{\lambda}{n}) \longrightarrow \mathcal{P}(\lambda)$$



A quote by Poisson:

Life is good for only two things: doing mathematics and teaching it!!

What did Poisson prove?

Fix $k \geq 0$. Then

$$\mathbb{P}(\mathrm{Bin}(n, \tfrac{\lambda}{n}) = k) \xrightarrow{n\to\infty} \mathbb{P}(\mathcal{P}(\lambda) = k)$$

$$\binom{n}{k} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \xrightarrow{n\to\infty} e^{-\lambda} \frac{\lambda^k}{k!}$$

check!

Example: drawing defective items (1)

Experiment:

- Item produced by a certain machine will be defective
  $\hookrightarrow$ with probability .1
- Sample of 10 items drawn

Question:
Probability that the sample contains at most 1 defective item

Model

- Success : we pick a defective item

  ↳ $p = 0.1$

- # Trials : we pick 10 items

  ↳ $n = 10$

X = # defective items

$\Rightarrow \boxed{X \sim Bin(10, 0.1)}$

We have $X \sim \text{Bin}(10, 0.1)$

We wish to compute

$$\boxed{\mathbb{P}(X \leq 1)} = \mathbb{P}(X=0) + \mathbb{P}(X=1)$$

$$= \underbrace{\binom{10}{0}}_{1} \underbrace{(0.1)^0}_{=1} (0.9)^{10} + \underbrace{\binom{10}{1}}_{10} (0.1)^1 (0.9)^9$$

$$\boxed{= .7361}$$

Poisson approximation. We have

$n = 10,\ p = 0.1 \Rightarrow np = 1 \leq 5,\ p \leq 0.1$

$\Rightarrow \text{Bin}(n,p) \approx \mathcal{P}(1)$

Thus

$\boxed{\mathbb{P}(X \leq 1)} = \mathbb{P}(X=0) + \mathbb{P}(X=1)$

$\simeq e^{-1}\left(\frac{1^0}{0!} + \frac{1^1}{1!}\right) = 2e^{-1}$

$\boxed{\simeq .7358}$ → good approximation!

Example: drawing defective items (2)

Random variable: Let

$$X = \# \text{ of defective items}$$

Then

$$X \sim \text{Bin}(n, p), \quad \text{with} \quad n = 10,\ p = .1$$

Exact probability: We have to compute

$$\begin{aligned} \mathbf{P}(X \le 1) &= \mathbf{P}(X = 0) + \mathbf{P}(X = 1) \\ &= (0.9)^{10} + 10 \times 0.1 \times (0.9)^9 \\ &= .7361 \end{aligned}$$

Example: drawing defective items (3)

Approximation: We use

$$\mathrm{Bin}(10, .1) \simeq \mathcal{P}(1)$$

Approximate probability: We have to compute

$$\begin{aligned} \mathbf{P}(X \leq 1) &= \mathbf{P}(X = 0) + \mathbf{P}(X = 1) \\ &\simeq e^{-1}(1 + 1) \\ &= .7358 \end{aligned}$$